

Differential Equations, Associators, and Recurrences for Amplitudes

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Abstract

We provide new methods to straightforwardly obtain compact and analytic expressions for ϵ -expansions of functions appearing in both field and string theory amplitudes. An algebraic method is presented to explicitly solve for recurrence relations connecting different ϵ -orders of a power series solution in ϵ of a differential equation. This strategy generalizes the usual iteration by Picard's method. Our tools are demonstrated for generalized hypergeometric functions. Furthermore, we match the ϵ -expansion of specific generalized hypergeometric functions with the underlying Drinfeld associator with proper Lie algebra and monodromy representations. We also apply our tools for computing ϵ -expansions for solutions to generic first-order Fuchsian equations (Schlesinger system). Finally, we set up our methods to systematically get compact and explicit α' -expansions of tree-level superstring amplitudes to any order in α' .

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1 Introduction

Scattering amplitudes describe the interactions of physical states and play an important role to determine physical observables measurable at colliders. Moreover, the computation of perturbative scattering amplitudes is of considerable interest both in quantum field-theory and string theory to reveal their underlying hidden symmetry structure and mathematical framework. The latter specifies the module of functions describing the amplitudes. These functions typically depend on the data of the external particles like their momenta, masses and scales. In field-theory the amplitudes are given by Feynman integrals over loop momenta. The generic functions describing Feynman integrals are iterated integrals, elliptic functions and perhaps generalizations thereof. On the other hand, in string theory amplitudes are given by integrals over vertex operator positions and the moduli space of the underlying world-sheet describing the interaction process of string states.

Amplitudes in field-theory very often can be described by certain differential equations or systems thereof with a given initial value problem subject to physical conditions [1]. On the other hand, partial differential equations based on Lie algebras appear in the context of conformal field theory underlying the symmetries on the string world-sheet [2, 3]. The differential equations (of regular singular points or generic divisors) capture essential information on singular limits of the amplitudes. These features are directly related to properties of the underlying string world-sheet.

In field-theory at a given loop order amplitudes depend on the parameter ϵ describing the dimensional regularization of the underlying Feynman integral with the space-time dimension $D = 4 - 2\epsilon$ as regularization parameter. One is interested in the Laurent series in ϵ of a Feynman integral. On the other hand, perturbative string amplitudes depend on the string tension α' accounting for the infinitely many heavy string modes of masses $M_{\text{string}}^2 \sim \alpha'^{-1}$. In this case one is interested in their power series expansion w.r.t. α' . Each term in this α' -expansion is typically described by \mathbf{Q} -linear combinations of iterated integrals of the same weight multiplied by homogeneous polynomials in kinematic invariants [4, 5].

Although the field-theory parameter ϵ and string tension α' are of completely different origin the functional dependence of amplitudes w.r.t. to these parameters leads to qualitative similar structures in both field- and string theory. Therefore, in practice finding explicit and analytic results for power series expansions w.r.t. these parameters is of fundamental importance. Obtaining in a fully systematic way a closed and compact expression for a given order in ϵ or α' , which does not rely on its lower orders to be computed in advance and which is straightforwardly applicable, is one main task in this work.

The differential equations underlying the amplitude integral give rise to recurrence relations connecting different orders of ϵ of a power series Ansatz in ϵ . We present an algebraic method to systematically solve for such recurrence relations stemming from differential equations with non-linear coefficients, i.e. coefficients of different orders in ϵ . This procedure gives an explicit solution to the recurrence relation providing for each order in ϵ a closed, compact and analytic expression. This strategy generalizes the usual iteration by Picard's method, which for a given order in ϵ requires information on all lower orders.

The novelty and importance of our recurrence method can be summarized as follows:

- We can straightforwardly construct power series expansions in ϵ as solutions of higher order differential equations with coefficients polynomial in ϵ .
- We obtain analytic, explicit and compact expressions at each order in ϵ in terms of iterated integrals without the necessity to compute their lower orders.
- Generically multiple polylogarithms (MPLs) appear only *linearly* and not any powers thereof.
- We can find finite solutions to differential equations at their regular singular points. Singularities do not appear at any point in our calculations, thus no regularization is necessary.

The module of hypergeometric functions is ubiquitous both in computing tree-level string amplitudes [6] and in the evaluation of Feynman diagrams cf. e.g. [7]. Therefore, finding an efficient procedure to determine the α' - or ϵ -expansion of these functions is an important problem and will be addressed in this work. Their underlying higher order differential equations lead to recurrence relations, which we explicitly solve. Our method provides closed expressions for ϵ -expansions of this big family of functions.

The differential equations for generalized hypergeometric functions are Fuchsian differential equations with regular singular points at $0, 1$ and ∞ . Hence, by properly assigning the Lie algebra and monodromy representations of the Knizhnik–Zamolodchikov (KZ) equation their underlying fundamental solutions can be matched with the ϵ -expansion of specific generalized hypergeometric functions. More concretely, for $x = 1$ we obtain a connection between the ϵ -expansion and the underlying Drinfeld associator, while for generic x a relation is established between the ϵ -expansion and one fundamental solution of the underlying KZ equation. This way we obtain a very elegant way of casting the full ϵ -expansion of a generalized hypergeometric function into the form of the Drinfeld associator. We also set up and apply our tools for computing ϵ -expansions of solutions to generic first order Fuchsian equations (Schlesinger system) and give explicit solutions by solving the underlying recurrences.

The organization of this work is as follows. In section 2 we introduce and discuss linear homogeneous recurrence relations and their generic solutions. The latter depends on the initial values only. In section 3 after introducing some basics in generalized hypergeometric functions we introduce integral operators for iterated integrals and we set up the recurrence relations for a power series Ansatz in ϵ solving the underlying hypergeometric differential equation. Then, we explicitly solve for these recursion relations providing compact and closed expressions for any order in the ϵ -expansion of generalized hypergeometric functions. In section 4 we establish the connection between generalized hypergeometric functions and the Drinfeld associator. We first rewrite the hypergeometric differential equation as first order Fuchsian differential equation. The latter is matched with the Lie algebra and monodromy representation of the underlying KZ equation thereby relating its fundamental solutions to hypergeometric functions with specific boundary conditions at the regular singular points. Furthermore, we also discuss ϵ -expansions of solutions to generic first order Fuchsian equations and apply our technique to solve recurrence relations for this type of differential equations. In section 5 we apply our results to α' -expansions of open superstring amplitudes yielding explicit expressions for any order in α' . Finally, in

Section 6 we present some other applications of our results. Generically, the latter are not given in terms of a minimal basis of multiple zeta values (MZVs). As a consequence, cyclic symmetry in the kinematic invariants is non-trivially fulfilled. As a result general MZV identities are generated, e.g. the sum theorem and generalizations thereof. The functions, which enter the all-order expansions and the MZV identities follow from a combinatorial approach. They can be related to hypergeometric functions, binomial coefficients and the (generalized) Fibonacci numbers.

2 Recurrence relations

One essential step of this work are n -th order linear homogeneous recurrence relations

$$w_k = \sum_{\alpha=1}^n c_\alpha w_{k-\alpha} , \quad (2.1)$$

with constant coefficients c_i and initial values $w_k = \bar{w}_k$ for $k = 0, 1, \dots, n-1$. In general the coefficients do not commute: $c_i c_j \neq c_j c_i$.

In subsection 2.1 useful definitions and notations for non-commutative coefficients will be introduced. The solution, i.e. a formula that expresses all w_k ($k \geq n$) in terms of the initial values \bar{w}_k only, will be presented in subsection 2.2.

2.1 The generalized operator product

A simple example is the following second order recurrence relation

$$w_k = c_1 w_{k-1} + c_2 w_{k-2} , \quad (2.2)$$

with initial values $w_0 = 1$ and $w_1 = c_1$. For $k = 5$ this gives:

$$w_5 = c_1^5 + c_1^3 c_2 + c_1^2 c_2 c_1 + c_1 c_2 c_1^2 + c_2 c_1^3 + c_1 c_2^2 + c_2 c_1 c_2 + c_2^2 c_1 . \quad (2.3)$$

It can be related to the integer partitions of 5, which use only 2 and 1:

$$5 = 1 + 1 + 1 + 1 + 1 = 2 + 1 + 1 + 1 = 2 + 2 + 1 . \quad (2.4)$$

Denoting how often 1 appears in a partition by j_1 and the number of 2's by j_2 , then each of these three partitions can be identified by a product $c_1^{j_1} c_2^{j_2}$, which for the case of interest are c_1^5 , $c_1^3 c_2$ and $c_1 c_2^2$. All these terms appear on the r.h.s. of (2.3). The remaining terms in (2.3) are permutations of these three products. Let us introduce the bracket $\{c_1^{j_1}, c_2^{j_2}\}$ for the sum of all possible distinct permutations of factors c_i , each one appearing j_i times ($i = 1, 2$). For example $j_1 = 1, j_2 = 2$ yields the following sum of three products:

$$\{c_1, c_2^2\} = c_1 c_2^2 + c_2 c_1 c_2 + c_2^2 c_1 . \quad (2.5)$$

With this bracket we can now write w_5 more compact as:

$$w_5 = \sum_{j_1+2j_2=5} \{c_1^{j_1}, c_2^{j_2}\} . \quad (2.6)$$

The sum over non-negative integers j_1 and j_2 represents all the integer partitions. It turns out, that the generalization of (2.6) solves the recurrence relation (2.2):

$$w_k = \sum_{j_1+2j_2=k} \{c_1^{j_1}, c_2^{j_2}\} . \quad (2.7)$$

Before the solution of the more general recurrence relation (2.1) and its proof are discussed, in the following subsection a proper definition and some basic properties of a generalized version of the bracket $\{c_1^{j_1}, c_2^{j_2}\}$ is given.

2.1.1 Definition

The object

$$\{c_1^{j_1}, c_2^{j_2}, \dots, c_n^{j_n}\} \quad (2.8)$$

is defined as the sum of all the

$$\left(\sum_{\alpha=1}^n j_\alpha \right) \quad (2.9)$$

possible distinct permutations of non-commutative factors c_i , each one appearing j_i times (with $j_i \in \mathbf{N} \cup \{0\}$, $i = 1, 2, \dots, n$). The non-negative integers j_i are referred to indices and the factors c_i to arguments of the generalized operator product (2.8). For example:

$$\{c_1, c_2\} = c_1 c_2 + c_2 c_1 , \quad (2.10)$$

$$\begin{aligned} \{c_1^2, c_2, c_3\} &= c_1^2 c_2 c_3 + c_1 c_2 c_1 c_3 + c_1 c_2 c_3 c_1 + c_2 c_1^2 c_3 + c_2 c_1 c_3 c_1 + c_2 c_3 c_1^2 \\ &+ c_1^2 c_3 c_2 + c_1 c_3 c_1 c_2 + c_1 c_3 c_2 c_1 + c_3 c_1^2 c_2 + c_3 c_1 c_2 c_1 + c_3 c_2 c_1^2 . \end{aligned} \quad (2.11)$$

For the case of two arguments the object (2.8) was used in [8] to solve a second order recurrence relation with non-commutative coefficients. There is a useful recursive definition¹ for (2.8) as:

$$\{c_1^{j_1}, c_2^{j_2}, \dots, c_n^{j_n}\} = \sum_{\substack{\alpha=1 \\ j_\alpha \neq 0}}^n c_\alpha \{c_1^{j_1}, c_2^{j_2}, \dots, c_\alpha^{j_\alpha-1}, \dots, c_n^{j_n}\} + \prod_{\beta=1}^n \delta_{0j_\beta} . \quad (2.12)$$

The product of Kronecker deltas gives a non-vanishing contribution in case all indices j_1, \dots, j_n are zero. Without this product an inconsistency would occur: for $j_1 = \dots = j_n = 0$ the sum on the r.h.s. of (2.12) becomes zero, while the l.h.s. should be one. Furthermore we have²:

$$\begin{aligned} \{c_1^0, c_2^{j_2}, \dots, c_n^{j_n}\} &= \{c_2^{j_2}, \dots, c_n^{j_n}\} , \\ \{c^j\} &= c^j , \end{aligned} \quad (2.13)$$

and especially $\{c^0\} = 1$.

¹The same formula with c_α to the right of the generalized operator product holds as well.

²Since in (2.8) the order of the arguments is irrelevant, we write identities, such as the first line of (2.13), without loss of generality for the first arguments only.

The definition (2.12) together with eqs. (2.13) allows to decrease step by step the indices and the number of arguments. This way (2.8) can be written in terms of non-commutative products. For instance applying (2.12) twice to all generalized operator products on the l.h.s. of (2.11) yields:

$$\begin{aligned} \{c_1^2, c_2, c_3\} &= c_1\{c_1, c_2, c_3\} + c_2\{c_1^2, c_3\} + c_3\{c_1^2, c_2\} \\ &= c_1^2\{c_2, c_3\} + c_1c_2\{c_1, c_3\} + c_1c_3\{c_1, c_2\} + c_2c_1\{c_1, c_3\} + c_2c_3c_1^2 \\ &\quad + c_3c_1\{c_1, c_2\} + c_3c_2c_1^2 . \end{aligned} \quad (2.14)$$

Applying (2.12) once again or using (2.10) gives the r.h.s. of (2.11). The definition of (2.8) can be extended to negative integer indices as:

$$\{c_1^{j_1}, c_2^{j_2}, \dots, c_n^{j_n}\} = 0 , \quad j_1 < 0 . \quad (2.15)$$

This extension turns out to be useful, when the indices of generalized operator products include summation indices. It allows to reduce the conditions for the summation regions. E.g. the condition $j_\alpha \neq 0$ in the sum on the r.h.s. of (2.12) can be dropped with this extension.

To prove that (2.12) gives indeed all distinct permutations, it is sufficient to show that:

1. the number of terms equals (2.9),
2. there are no identical terms
3. and all terms contain each non-commutative factor c_i exactly j_i times.

The third point is obviously fulfilled. The second one is also quite obvious, since every summand of the sum in (2.12) starts with a different factor. Using the definition again yields, that all terms coming from the same summand and therefore have the same first factor, have a different second factor and so on. The first point is also true, since the number of terms on the r.h.s. of (2.12) is

$$\sum_{\alpha=1}^n \binom{-1 + \sum_{\beta=1}^n j_\beta}{j_1, j_2, \dots, j_\alpha - 1, \dots, j_n} . \quad (2.16)$$

The above expression can easily be transformed into (2.9) using the definition of the multinomial coefficient in terms of factorials.

The generalized operator product (2.8) is closely related to the shuffle product:

$$\{c_1^{j_1}, c_2^{j_2}, \dots, c_n^{j_n}\} = \underbrace{c_1 \dots c_1}_{j_1} \sqcup \underbrace{c_2 \dots c_2}_{j_2} \sqcup \dots \sqcup \underbrace{c_n \dots c_n}_{j_n} . \quad (2.17)$$

However, the notation on the l.h.s. is more compact, in particular for the applications in the following sections.

2.1.2 Basic properties

With the definition (2.12) the following basic properties can easily be proven. Factors a which commute with all arguments, i.e. $c_i a = a c_i$ can be factorized:

$$\{(ac_1)^{j_1}, c_2^{j_2}, \dots, c_n^{j_n}\} = a^{j_1} \{c_1^{j_1}, c_2^{j_2}, \dots, c_n^{j_n}\} . \quad (2.18)$$

Identical arguments can be combined as:

$$\{c_1^{j_1}, c_1^{j_2}, c_3^{j_3}, \dots, c_n^{j_n}\} = \{c_1^{j_1+j_2}, c_3^{j_3}, \dots, c_n^{j_n}\} \binom{j_1+j_2}{j_1} . \quad (2.19)$$

The binomial coefficient³ ensures, that the number of terms is the same on both sides. While sums can be treated according to

$$\{c_1 + c_2, c_3^{j_3}, \dots, c_n^{j_n}\} = \{c_1, c_3^{j_3}, \dots, c_n^{j_n}\} + \{c_2, c_3^{j_3}, \dots, c_n^{j_n}\} , \quad (2.20)$$

one has to be careful, when such arguments appear with exponents greater than one. Before these cases are discussed, note that the generalized operator product can be used for a generalized version of the binomial theorem, which is also valid for non-commutative quantities c_1 and c_2 :

$$(c_1 + c_2)^j = \sum_{j_1+j_2=j} \{c_1^{j_1}, c_2^{j_2}\} , \quad (2.21)$$

with non-negative integers j , j_1 and j_2 . Applying naively this relation to arguments of (2.8) leads to inconsistencies. E.g.:

$$\begin{aligned} \{(c_1 + c_2)^2, c_3\} &\stackrel{?}{=} \left\{ \sum_{\alpha=0}^2 \{c_1^{2-\alpha}, c_2^{\alpha}\}, c_3 \right\} \\ &= \{c_1^2, c_3\} + \{c_2^2, c_3\} + \{c_1 c_2, c_3\} + \{c_2 c_1, c_3\} . \end{aligned} \quad (2.22)$$

Eq. (2.20) is used in the last step. Using instead the definition (2.12), gives:

$$\begin{aligned} \{(c_1 + c_2)^2, c_3\} &= (c_1 + c_2)(c_1 + c_2)c_3 + (c_1 + c_2)c_3(c_1 + c_2) \\ &\quad + c_3(c_1 + c_2)(c_1 + c_2) . \end{aligned} \quad (2.23)$$

Comparing (2.22) and (2.23) shows that $c_1 c_3 c_2 + c_2 c_3 c_1$ is missing in (2.22). To avoid this problem, one simply has to ignore the inner curly brackets when applying (2.21) to arguments of (2.8). Hence, the following relation is consistent:

$$\{(c_1 + c_2)^j, c_3^{j_3}, \dots, c_n^{j_n}\} = \sum_{j_1+j_2=j} \{c_1^{j_1}, c_2^{j_2}, c_3^{j_3}, \dots, c_n^{j_n}\} . \quad (2.24)$$

This can be easily generalized to multinomials:

$$\{(c_1 + c_2 + \dots + c_n)^j, c_{n+1}^{j_{n+1}}, \dots\} = \sum_{j_1+j_2+\dots+j_n=j} \{c_1^{j_1}, c_2^{j_2}, \dots, c_n^{j_n}, c_{n+1}^{j_{n+1}}, \dots\} . \quad (2.25)$$

Besides these basic properties, there are more intricate identities satisfied by generalized operator products. They will be discussed in section 6.

³In order not to conflict with (2.15), we use $\binom{j_1+j_2}{j_1} = 0$ for $(j_1 < 0) \vee (j_2 < 0)$.

2.2 Solution

The n -th order linear homogeneous recurrence relation (2.1) is solved by:

$$w_k = \sum_{\alpha=0}^{n-1} \sum_{j_1+2j_2+\dots+nj_n=k-n-\alpha} \{c_1^{j_1}, c_2^{j_2}, \dots, c_n^{j_n}\} \sum_{\beta=\alpha+1}^n c_\beta \bar{w}_{n-\beta+\alpha}, \quad k \geq n. \quad (2.26)$$

Note, that the r.h.s. of (2.26) only contains initial values \bar{w}_l . The second sum is over n -tuples of non-negative integers j_1, \dots, j_n which solve the equation:

$$\sum_{\gamma=1}^n \gamma j_\gamma = k - n - \alpha. \quad (2.27)$$

In the following we shall prove by induction that (2.26) solves (2.1). The regions $2n > k \geq n$ and $k \geq 2n$ are discussed separately. The first region is required to prove the base case $k = 2n$ of the induction for $k \geq 2n$. The induction for $2n > k \geq n$ has the two base cases $k = n$ and $k = n+1$. In the first case $k = n$, the only non-zero contribution comes from $\alpha = j_1 = j_2 = \dots = j_n = 0$:

$$w_n = \sum_{\beta=1}^n c_\beta \bar{w}_{n-\beta}. \quad (2.28)$$

The second case $k = n+1$ has two parts. One with $\alpha = 1$, $j_1 = j_2 = \dots = j_n = 0$ and the other with $j_1 = 1$, $\alpha = j_2 = \dots = j_n = 0$:

$$\begin{aligned} w_{n+1} &= c_1 \sum_{\gamma=1}^n c_\gamma \bar{w}_{n-\gamma} + \sum_{\beta=2}^n c_\beta \bar{w}_{n+1-\beta} \\ &= c_1 w_n + \sum_{\beta=2}^n c_\beta \bar{w}_{n+1-\beta} \\ &= \sum_{\beta=1}^n c_\beta w_{n+1-\beta}. \end{aligned} \quad (2.29)$$

Eq. (2.28) is used in the second line of (2.29). Both cases (2.28) and (2.29) are in agreement with the eq. (2.1) and the initial conditions. The recursive definition (2.12) of the generalized operator product is particularly useful for the inductive step:

$$\begin{aligned} w_k &= \sum_{\alpha=0}^{n-1} \sum_{j_1+2j_2+\dots+nj_n=k-n-\alpha} \sum_{\gamma=1}^n c_\gamma \{c_1^{j_1}, \dots, c_\gamma^{j_\gamma-1}, \dots, c_n^{j_n}\} \sum_{\beta=\alpha+1}^n c_\beta \bar{w}_{n-\beta+\alpha} \\ &\quad + \sum_{\alpha=0}^{n-1} \sum_{j_1+2j_2+\dots+nj_n=k-n-\alpha} \prod_{\gamma=1}^n \delta_{0j_\gamma} \sum_{\beta=\alpha+1}^n c_\beta \bar{w}_{n-\beta+\alpha}. \end{aligned} \quad (2.30)$$

Shifting $j_\gamma \rightarrow j_\gamma + 1$ on the r.h.s. of the first line gives:

$$\sum_{\gamma=1}^n c_\gamma \sum_{\alpha=0}^{n-1} \sum_{j_1+2j_2+\dots+nj_n=k-n-\alpha-\gamma} \{c_1^{j_1}, \dots, c_n^{j_n}\} \sum_{\beta=\alpha+1}^n c_\beta \bar{w}_{n-\beta+\alpha}$$

$$\begin{aligned}
&= \sum_{\gamma=1}^n c_{\gamma} \cdot \begin{cases} w_{k-\gamma} & \text{for } k - \gamma \geq n \\ 0 & \text{else.} \end{cases} \\
&= \sum_{\gamma=1}^{\min(n, k-n)} c_{\gamma} w_{k-\gamma} .
\end{aligned} \tag{2.31}$$

The second line of (2.30) is non-zero only if there is a solution for $k - n - \alpha = 0$. Inserting the upper bound $\alpha \leq n - 1$ of the first sum, gives the condition $k < 2n$. Combining both lines of (2.30) for this region yields

$$w_k = \sum_{\gamma=1}^{k-n} c_{\gamma} w_{k-\gamma} + \sum_{\beta=k-n+1}^n c_{\beta} \bar{w}_{k-\beta} , \tag{2.32}$$

which is identical to (2.1). In the region $k \geq 2n$ the second line of (2.30) becomes zero and the upper bound in (2.31) is n , since $k - n \geq n$. This also results in (2.1).

Finally, it is easy to prove that the inhomogeneous recurrence relation ($k \geq n$)

$$w_k^{(\text{inh})} = \sum_{\alpha=1}^n c_{\alpha} w_{k-\alpha}^{(\text{inh})} + d_k \tag{2.33}$$

is solved by

$$w_k^{(\text{inh})} = w_k + \sum_{\alpha=n}^k \sum_{j_1+2j_2+\dots+nj_n=k-\alpha} \{c_1^{j_1}, c_2^{j_2}, \dots, c_n^{j_n}\} d_{\alpha} , \tag{2.34}$$

where w_k is the solution of the corresponding homogeneous recurrence relation and d_k is a inhomogeneity, that depends on k .

3 Expansion of generalized hypergeometric functions

The generalized Gauss function or generalized hypergeometric function ${}_pF_{p-1}$ is given by the power series [9]

$${}_pF_{p-1}(\vec{a}; \vec{b}; z) \equiv {}_pF_{p-1} \left[\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_{p-1} \end{matrix} ; z \right] = \sum_{m=0}^{\infty} \frac{\prod_{i=1}^p (a_i)^m}{\prod_{j=1}^{p-1} (b_j)^m} \frac{z^m}{m!} , \quad p \geq 1 , \tag{3.1}$$

with some parameters $a_i, b_j \in \mathbf{R}$ and with the Pochhammer (rising factorial) symbol:

$$(a)^n = \frac{\Gamma(a+n)}{\Gamma(a)} = a(a+1) \dots (a+n-1) .$$

The series (3.1) enjoys the p -th order differential equation (with $y = {}_pF_{p-1}$)

$$\theta (\theta + b_1 - 1) (\theta + b_2 - 1) \dots (\theta + b_{p-1} - 1) y - z (\theta + a_1) (\theta + a_2) \dots (\theta + a_p) y = 0 , \quad (3.2)$$

with the differential operator:

$$\theta = z \frac{d}{dz} . \quad (3.3)$$

Furthermore, the function (3.1) satisfies:

$$\begin{aligned} (\theta + a_i) {}_pF_{p-1}(\vec{a}; \vec{b}; z) &= a_i {}_pF_{p-1} \left[\begin{matrix} a_1, \dots, a_i + 1, \dots, a_p \\ b_1, \dots, b_{p-1} \end{matrix}; z \right] , \\ (\theta + b_j - 1) {}_pF_{p-1}(\vec{a}; \vec{b}; z) &= (b_j - 1) {}_pF_{p-1} \left[\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_j - 1, \dots, b_{p-1} \end{matrix}; z \right] , \\ \frac{d}{dz} {}_pF_{p-1}(\vec{a}; \vec{b}; z) &= \frac{\prod_{i=1}^p a_i}{\prod_{j=1}^{p-1} b_j} {}_pF_{p-1} \left[\begin{matrix} a_1 + 1, \dots, a_p + 1 \\ b_1 + 1, \dots, b_{p-1} + 1 \end{matrix}; z \right] . \end{aligned} \quad (3.4)$$

The ϵ - or α' -expansion of a given generalized hypergeometric function

$${}_pF_{p-1} \left[\begin{matrix} m_1 + \alpha' a_1, \dots, m_p + \alpha' a_p \\ n_1 + \alpha' b_1, \dots, n_{p-1} + \alpha' b_{p-1} \end{matrix}; z \right] , \quad m_i, n_j \in \mathbf{Z} \quad (3.5)$$

is expressible in terms of generalized polylogarithms with coefficients, that are ratios of polynomials. For each p it is sufficient to derive the expansion for one set of integers \vec{m} and \vec{n} only. By using eqs. (3.4) any function ${}_pF_{p-1}(\vec{a}; \vec{b}; z)$ can be expressed as a linear combination of other functions ${}_pF_{p-1}(\vec{m} + \vec{a}; \vec{n} + \vec{b}; z)$ with parameters that differ from the original ones by an integer shift and the first $p - 1$ derivatives thereof [10].

In this section we shall present and apply our new technique to solve for recurrences to compute the α' -expansion of (3.5). To warm up and for completeness we begin with the case of the $p = 2$ hypergeometric function:

$${}_2F_1 \left[\begin{matrix} -\alpha' a, \alpha' b \\ 1 + \alpha' b \end{matrix}; z \right] = \sum_{k=0}^{\infty} u_k(z) (\alpha')^k . \quad (3.6)$$

However, we shall mainly be concerned with the case $p = 3$. For the latter the expansion takes the form:

$${}_3F_2 \left[\begin{matrix} \alpha' a_1, \alpha' a_2, \alpha' a_3 \\ 1 + \alpha' b_1, 1 + \alpha' b_2 \end{matrix}; z \right] = \sum_{k=0}^{\infty} v_k(z) (\alpha')^k . \quad (3.7)$$

Finally, the general case p will be elaborated at the end of this section.

To obtain all-order expressions for $v_k(z)$ and $u_k(z)$ the differential equations in z for the functions (3.5) are used. Combining eqs. (3.4) gives the differential equation (3.2). Inserting

the expansions (3.6) and (3.7) into (3.2) leads to the following recursive differential equations for $u_k(z)$ and $v_k(z)$, respectively:

$$0 = (z-1) \theta^2 u_k(z) + [z(b-a) - b] \theta u_{k-1}(z) - z ab u_{k-2}(z) , \quad (3.8)$$

$$0 = (z-1) \theta^3 v_k(z) + [z(a_1 + a_2 + a_3) - b_1 - b_2] \theta^2 v_{k-1}(z) \\ + [z(a_1 a_2 + a_2 a_3 + a_3 a_1) - b_1 b_2] \theta v_{k-2}(z) + z a_1 a_2 a_3 v_{k-3}(z) . \quad (3.9)$$

To solve these equations we transform them into recurrence relations. This is achieved by replacing the derivatives and integrations in the formal solution of these differential equations by differential and integral operators, respectively. In addition, the boundary conditions have to be respected. The recurrence relations have the form (2.1), where the non-commutative coefficients c_i represent differential and integral operators.

Applying recurrences to expand hypergeometric functions was first proposed in the field-theory context in [11, 12]. In [13] the recurrence relations have been used to calculate the expansions (3.7) and (3.6) order by order. However, with the general solution (2.26) the all-order expansions can now systematically be constructed and straightforwardly be given in closed form.

In the next subsection we shall discuss our notation. Then, in the subsequent subsections we shall present and solve the recurrence relations for $u_k(z)$, $v_k(z)$ and for general p .

3.1 Integral operators, multiple polylogarithms and multiple zeta values

We introduce the integral operators

$$I(1) f(z) = \int_0^z \frac{dt}{1-t} f(t) , \quad (3.10)$$

$$\text{and } I(0) f(z) = \int_0^z \frac{dt}{t} f(t)$$

to define the following multiple integral operator

$$I(\underbrace{0, \dots, 0}_{n_1-1}, 1, \dots, \underbrace{0, \dots, 0}_{n_d-1}, 1) := I(0)^{n_1-1} I(1) I(0)^{n_2-1} I(1) \dots I(0)^{n_d-1} I(1) , \quad (3.11)$$

with the multiple index $\vec{n} = (n_1, n_2, \dots, n_d)$. Acting with the operator (3.11) on⁴ the constant function 1 yields MPLs:

$$I(\underbrace{0, \dots, 0}_{n_1-1}, 1, \dots, \underbrace{0, \dots, 0}_{n_d-1}, 1) 1 = \mathcal{L}i_{\vec{n}}(z) \equiv \mathcal{L}i_{n_1, \dots, n_d}(z, \underbrace{1, \dots, 1}_{d-1}) = \sum_{0 < k_d < \dots < k_1} \frac{z^{k_1}}{k_1^{n_1} \cdot \dots \cdot k_d^{n_d}} . \quad (3.12)$$

⁴In the sequel we will not write this 1.

For $z = 1$ the MPLs become MZVs

$$\zeta(\vec{n}) = \mathcal{L}i_{\vec{n}}(1) = I(\underbrace{0, \dots, 0}_{n_1-1}, 1, \dots, \underbrace{0, \dots, 0}_{n_d-1}, 1) \Big|_{z=1} , \quad (3.13)$$

with the following definition of MZVs:

$$\zeta(\vec{n}) \equiv \zeta(n_1, \dots, n_d) = \sum_{0 < k_d < \dots < k_1} k_1^{-n_1} \cdot \dots \cdot k_d^{-n_d} . \quad (3.14)$$

Both for MPLs and for MZVs the weight w is defined as the sum of all indices:

$$w = n_1 + n_2 + \dots + n_d . \quad (3.15)$$

Using the representations in terms of integral operators the weight is equivalent to the total number of integral operators. The depth d is defined as the number of indices, i.e.:

$$d = \dim(\vec{n}) . \quad (3.16)$$

In terms of integral operators, this is the number of operators $I(1)$. For example

$$I(0)I(1)I(1) = \mathcal{L}i_{2,1}(z) \xrightarrow{z=1} \zeta(2, 1) \quad (3.17)$$

are weight $w = 3$ and depth $d = 2$ MPLs and MZVs, respectively. It is useful to combine both products of integral operators (3.10) and the differential operator θ defined in (3.3) into the shorter form:

$$I(p_1)I(p_2) \dots I(p_n) = I(p_1, p_2, \dots, p_n) , \quad p_i \in \{0, 1, \theta\} , \quad I(\theta) \equiv \theta . \quad (3.18)$$

Up to boundary values the differential operator θ is the inverse of $I(0)$:

$$I(\theta, 0)f(z) = I(0, \theta)f(z) = f(z) . \quad (3.19)$$

The results of the following sections will often contain sums of the form:

$$\sum_{\dots} \zeta(\vec{n}) . \quad (3.20)$$

Above, the dots may represent conditions for the weight w , the depth d , specific indices n_i or other quantities referring to the MZVs $\zeta(\vec{n})$ in the sum. The sum runs over all sets of positive integers \vec{n} , that satisfy these requirements. It is understood that $n_1 > 1$. For example the sum of all MZVs of weight $w = 5$ and depth $d = 2$ is represented as:

$$\sum_{\substack{w=5 \\ d=2}} \zeta(\vec{n}) = \zeta(4, 1) + \zeta(3, 2) + \zeta(2, 3) . \quad (3.21)$$

Further conditions could include the first index n_1 or the number of indices d_1 , which equal one:

$$\sum_{\substack{w=5; \ d=2 \\ n_1 \geq 3; \ d_1=0}} \zeta(\vec{n}) = \zeta(3, 2) . \quad (3.22)$$

Obviously $d_1 = 0$ is equivalent to $n_i \geq 2$ ($i = 1, \dots, d$). More general d_i is defined as the number of indices, which equal i , so that $d = \sum_i d_i$. In some cases a weighting ω is included, which can depend on the indices \vec{n} or other quantities. For example the following sum has $\omega = d_1$:

$$\sum_{\substack{w=6 \\ d=3}} \zeta(\vec{n}) d_1 = 2\zeta(4, 1, 1) + \zeta(3, 2, 1) + \zeta(3, 1, 2) + \zeta(2, 3, 1) + \zeta(2, 1, 3) . \quad (3.23)$$

In our notation the well known sum theorem [14] reads

$$\sum_{\substack{w=a \\ d=b}} \zeta(\vec{n}) = \zeta(a) , \quad (3.24)$$

which means, that for given weight and depth the sum of all MZVs equals the single zeta value (depth one MZV) of that weight (independent of the given depth). The same notation is used for MPLs.

Some sums use multiple indices $\vec{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_a)$:

$$\sum_{\vec{\alpha}} f(\alpha_1, \alpha_2, \dots, \alpha_a) = \sum_{\alpha_1} \sum_{\alpha_2} \dots \sum_{\alpha_a} f(\alpha_1, \alpha_2, \dots, \alpha_a) . \quad (3.25)$$

In this context it is necessary to distinguish between functions $f(\alpha_1, \alpha_2, \dots, \alpha_a)$, which depend on elements of multi-indices, and functions $g(\vec{\alpha})$, which have multi-indices as arguments:

$$\sum_{\vec{\alpha}} g(\vec{\alpha}) = \sum_{\alpha_1} g(\alpha_1) \sum_{\alpha_2} g(\alpha_2) \dots \sum_{\alpha_a} g(\alpha_a) . \quad (3.26)$$

The latter only occur in combination with multi-index sums and they represent functions $g(\alpha_i)$, which have only one element as argument. The summation regions for the elements α_i follow from the summation region of the vector $\vec{\alpha}$ in a natural way. All indices of these sums are non-negative integers ($\alpha_i \geq 0$). The sum of all elements of a multi-index $\vec{\alpha}$ is written as $|\vec{\alpha}| = \alpha_1 + \alpha_2 + \dots + \alpha_d$. This notation is especially used for weightings in sums of MZVs (3.20). It is understood, that the number of elements of the multi-indices equals the depth of the corresponding MZVs.

3.2 Second order recurrence relation

As a warm up in this subsection the solution of the recurrence relation for the coefficients $u_k(z)$ of the expansion (3.6) is calculated. This result is already known [15]. The relation reads

$$u_k(z) = c_1 u_{k-1}(z) + c_2 u_{k-2}(z) , \quad (3.27)$$

with

$$\begin{aligned} c_1 &= -aI(0, 1, \theta) - bI(0) , \\ c_2 &= -abI(0, 1) , \end{aligned} \quad (3.28)$$

and the initial values $u_0 = 1$ and $u_1 = 0$. According to (2.26) the solution is:

$$u_k(z) = -ab \sum_{j_1+2j_2=k-2} \{(-aI(0, 1, \theta) - bI(0))^{j_1}, (-abI(0, 1))^{j_2}\} I(0, 1) . \quad (3.29)$$

Eq. (3.12) implies, that the final expression only contains the integral operators $I(0)$ and $I(1)$. Therefore the first step should be to eliminate the differential operator $I(\theta)$. This is achieved by the relation (3.19) and the following identity:

$$\{I(0, \vec{p}_1, \theta)^{j_1}, I(0, \vec{p}_2, \theta)^{j_2}, \dots, I(0, \vec{p}_n, \theta)^{j_n}\} = I(0) \{I(\vec{p}_1)^{j_1}, I(\vec{p}_2)^{j_2}, \dots, I(\vec{p}_n)^{j_n}\} I(\theta) . \quad (3.30)$$

The removal of $I(\theta)$ works because every argument starts with an $I(0)$ and ends with an $I(\theta)$. The vectors \vec{p}_i are arbitrary sequences of the elements $\{0, 1, \theta\}$. With the relations (2.18), (2.25) and (3.30) the result (3.29) can be transformed to:

$$u_k(z) = \sum_{\alpha=1}^{k-1} (-1)^{k+1} a^{k-\alpha} b^\alpha \sum_{\beta} (-1)^\beta I(0) \{I(1)^{k-\alpha-1-\beta}, I(0)^{\alpha-1-\beta}, I(1, 0)^\beta\} I(1) . \quad (3.31)$$

An identity, which will be discussed in section 6, allows to simplify the generalized operator product and the sum over β to arrive at:

$$u_k(z) = \sum_{\alpha=1}^{k-1} (-1)^{k+1} a^{k-\alpha} b^\alpha I(0)^\alpha I(1)^{k-\alpha} = \sum_{\alpha=1}^{k-1} (-1)^{k+1} a^{k-\alpha} b^\alpha \mathcal{L}i_{(\alpha+1, \{1\}^{k-\alpha-1})}(z) . \quad (3.32)$$

In the final step eq. (3.12) has been used to express the result in terms of MPLs. Therefore, the hypergeometric function (3.6) can be written as:

$${}_2F_1 \left[\begin{matrix} -\alpha' a, & \alpha' b \\ 1 + \alpha' b \end{matrix} ; z \right] = 1 - \sum_{k=2}^{\infty} (-\alpha')^k \sum_{\alpha=1}^{k-1} a^{k-\alpha} b^\alpha \mathcal{L}i_{(\alpha+1, \{1\}^{k-\alpha-1})}(z) . \quad (3.33)$$

Of particular interest is the case $z = 1$, since the resulting object arises in the four-point open string amplitude [4]. This will be discussed in subsection 5.1 and further discussions will follow there.

3.3 Third order recurrence relation

The recurrence relation for the coefficients $v_k(z)$ of the series (3.7), which has been derived in [13], reads

$$v_k(z) = c_1 v_{k-1}(z) + c_2 v_{k-2}(z) + c_3 v_{k-3}(z) , \quad (3.34)$$

with

$$\begin{aligned} c_1 &= \Delta_1 I(0, 0, 1, \theta, \theta) - Q_1 I(0) , \\ c_2 &= \Delta_2 I(0, 0, 1, \theta) - Q_2 I(0, 0) , \\ c_3 &= \Delta_3 I(0, 0, 1) , \end{aligned} \tag{3.35}$$

and $v_0 = 1$; $v_1, v_2 = 0$ as initial values. Furthermore, we define:

$$\begin{aligned} \Delta_1 &= a_1 + a_2 + a_3 - b_1 - b_2 , \\ \Delta_2 &= a_1 a_2 + a_2 a_3 + a_3 a_1 - b_1 b_2 , \\ \Delta_3 &= a_1 a_2 a_3 , \\ Q_1 &= b_1 + b_2 , \\ Q_2 &= b_1 b_2 . \end{aligned} \tag{3.36}$$

According to (2.26) the solution of (3.34) is:

$$v_k(z) = \sum_{j_1+2j_2+3j_3=k-3} \{c_1^{j_1}, c_2^{j_2}, c_3^{j_3}\} c_3 . \tag{3.37}$$

Applying the definitions (3.35) as well as the identities (2.18), (2.25), (3.19) and (3.30) gives:

$$\begin{aligned} v_k(z) &= \sum_{m_1+l_1+2(l_2+m_2)+3m_3=k-3} (-1)^{l_1+l_2} \Delta_1^{m_1} \Delta_2^{m_2} \Delta_3^{m_3+1} Q_1^{l_1} Q_2^{l_2} \\ &\quad \times I(0, 0) \{I(0)^{l_1}, I(0, 0)^{l_2}, I(1)^{m_1}, I(1, 0)^{m_2}, I(1, 0, 0)^{m_3}\} I(1) . \end{aligned} \tag{3.38}$$

This compact expression allows to extract any order of the expansion of the hypergeometric function (3.7) without having to calculate lower orders. For Mathematica implementations the routine 'DistinctPermutations' is useful to evaluate the generalized operator products. With this the iterated integrals can directly be written in terms of MPLs or MZVs in the case $z = 1$. All functions are finite. As a consequence no singularities occur.

As already mentioned, expansions of other hypergeometric functions ${}_3F_2(\vec{m} + \vec{a}; \vec{n} + \vec{b}; z)$, can be obtained from the result (3.38) with the relations (3.4). Two such functions, which enter the five-point open superstring amplitude, are the topic of subsection 5.2.

As in the second order case there is an identity, which allows to remove the generalized operator product yielding a representation in terms of MPLs. How to get there will be discussed in section 6.

3.4 Recurrence relation at p -th order

Finally, let us discuss the case of general p , which assumes the expansion:

$${}_pF_{p-1} \left[\begin{matrix} \alpha' a_1, \dots, \alpha' a_p \\ 1 + \alpha' b_1, \dots, 1 + \alpha' b_{p-1} \end{matrix}; z \right] = \sum_{k=0}^{\infty} w_k^p(z) (\alpha')^k . \tag{3.39}$$

From the differential equation (3.2) for the coefficients in (3.39) we obtain the following recurrence relation [12]:

$$w_k^p(z) = \sum_{\alpha=1}^p c_\alpha^p w_{k-\alpha}^p(z) \quad , \quad k \geq p . \quad (3.40)$$

The initial conditions are $w_0^p(z) = 1$ and $w_k^p(z) = 0$ for $0 < k < p$. In (3.40) the coefficients are

$$c_\alpha^p = \Delta_\alpha^p I(0)^{p-1} I(1) \theta^{p-\alpha} - Q_\alpha^p I(0)^\alpha , \quad (3.41)$$

with

$$\Delta_\alpha^p = P_\alpha^p - Q_\alpha^p \quad , \quad \alpha = 1, \dots, p-1 \quad , \quad \Delta_p^p = P_p^p , \quad (3.42)$$

and P_α^p the α -th symmetric product (elementary symmetric function) of the parameters a_1, \dots, a_p and Q_β^p the β -th symmetric product of the parameters b_1, \dots, b_{p-1} , i.e.:

$$\begin{aligned} P_\alpha^p &= \sum_{\substack{i_1, \dots, i_\alpha=1 \\ i_1 < i_2 < \dots < i_\alpha}}^p a_{i_1} \cdot \dots \cdot a_{i_\alpha} \quad , \quad \alpha = 1, \dots, p , \\ Q_\beta^p &= \sum_{\substack{i_1, \dots, i_\beta=1 \\ i_1 < i_2 < \dots < i_\beta}}^{p-1} b_{i_1} \cdot \dots \cdot b_{i_\beta} \quad , \quad \beta = 1, \dots, p-1 \quad , \quad Q_p^p = 0 . \end{aligned} \quad (3.43)$$

According to (2.26) the solution follows from:

$$w_k^p(z) = \sum_{j_1+2j_2+\dots+pj_p=k-p} \{ (c_1^p)^{j_1}, (c_2^p)^{j_2}, \dots, (c_p^p)^{j_p} \} c_p^p . \quad (3.44)$$

Performing the same steps as for $p = 2$ and $p = 3$ leads to the following result:

$$\begin{aligned} w_k^p(z) &= \sum_{\vec{l}, \vec{m}} (-1)^{|\vec{l}|} (\Delta_1^p)^{m_1} (\Delta_2^p)^{m_2} \dots (\Delta_{p-1}^p)^{m_{p-1}} (\Delta_p^p)^{m_p+1} (Q_1^p)^{l_1} (Q_2^p)^{l_2} \dots (Q_{p-1}^p)^{l_{p-1}} \\ &\times I(0)^{p-1} \{ I(0)^{l_1}, \dots, I(\underbrace{0, \dots, 0}_{p-1})^{l_{p-1}}, I(1)^{m_1}, \dots, I(1, \underbrace{0, \dots, 0}_{p-1})^{m_p} \} I(1) . \end{aligned} \quad (3.45)$$

The sum is over the multi-indices $\vec{l} = (l_1, l_2, \dots, l_{p-1})$ and $\vec{m} = (m_1, m_2, \dots, m_p)$, which solve the equation:

$$\sum_{\alpha=1}^{p-1} \alpha(l_\alpha + m_\alpha) + pm_p = k - p . \quad (3.46)$$

Note, that in contrast to the findings in [12] the result (3.45) allows to express any order $w_k^p(z)$ directly without using lower orders. A representation in terms of MPLs will be given in section 6.

As a final comment we note that in the series (3.6), (3.7) and (3.39) the powers of α' are always accompanied by MPLs of uniform degree of transcendentality (maximal transcendentality of weight k). The transcendentality weight w is given by the degree of the MPL: $w(\ln x) = 1$, $w(\mathcal{L}i_a) = a$ and $w(\mathcal{L}i_{n_1, \dots, n_d}) = \sum_{i=1}^d n_i$. The degree of transcendentality for a product is defined to be the sum of the degrees of each factor. The maximal transcendentality in the power series expansions (3.6), (3.7) and (3.39) w.r.t. α' simply follows from the underlying recursion relations (3.31), (3.38) and (3.45) for their expansion coefficients $w_k(z)$.

4 Fuchsian equations and explicit solutions by iterations and recurrences

4.1 Generalized hypergeometric functions and Fuchsian equations

For $y = {}_{q+1}F_q(x)$, with $p = n = q + 1$ the differential equation (3.2) becomes

$$x^{n-1} (1-x) \frac{d^n y}{dx^n} + \mathfrak{a}_0 y + \sum_{\nu=1}^{n-1} x^{\nu-1} (\mathfrak{a}_\nu x - \mathfrak{b}_\nu) \frac{d^\nu y}{dx^\nu} = 0 , \quad (4.1)$$

with the parameters $\mathfrak{a}_i, \mathfrak{b}_j$ given as polynomials in a_r, b_s . This is a Fuchsian equation with regular singularities at $x = 0$, $x = 1$ and $x = \infty$ and n linearly independent solutions for $|x| < 1$ and $n - 1$ independent solutions at $x = 1$ to be specified below. Furthermore, with

$$\begin{aligned} f_1 &= y , \\ f_2 &= y' , \\ &\vdots \\ f_n &= y^{(n-1)} , \end{aligned} \quad (4.2)$$

eq. (4.1) can be brought into a system of first order linear differential equations

$$\partial_x \mathbf{f} = A(x) \mathbf{f} , \quad (4.3)$$

with some quadratic matrix A given by the parameters a_i, b_j . The resulting system (4.3) implies non-simple or spurious poles, which can be transformed away by a suitable transformation T :

$$\begin{aligned} B &= T^{-1} A T - T^{-1} \partial_x T \\ \mathbf{f} &= T \mathbf{g} . \end{aligned} \quad (4.4)$$

Eventually, eq. (4.1) can be cast into a Fuchsian system of first order

$$\frac{d\mathbf{g}}{dx} = \left(\frac{B_0}{x} + \frac{B_1}{1-x} \right) \mathbf{g} , \quad (4.5)$$

with regular singularities at $x = 0$, $x = 1$ and $x = \infty$. For $q = 1$ we have

$$\mathfrak{a}_0 = -a_1 a_2, \quad \mathfrak{a}_1 = -(1 + a_1 + a_2), \quad \mathfrak{b}_1 = -b_1 ,$$

$$A = \begin{pmatrix} 0 & 1 \\ \frac{a_1 a_2}{x} + \frac{a_1 a_2}{1-x} & \frac{1+a_1+a_2-b_1}{1-x} - \frac{b_1}{x} \end{pmatrix} ,$$

and with $T = \begin{pmatrix} 1 & 0 \\ 0 & x^{-1} \end{pmatrix}$ we obtain:

$$B_0 = \begin{pmatrix} 0 & 1 \\ 0 & \beta_1 \end{pmatrix} , \quad B_1 = \begin{pmatrix} 0 & 0 \\ \alpha_0 & \alpha_1 + \beta_1 \end{pmatrix} ,$$

with α_i being the $2-i$ -th elementary symmetric function of a_1, a_2 and β_i the $2-i$ -th elementary symmetric function of $1 - b_1$, i.e.

$$\begin{aligned}\alpha_0 &= a_1 a_2 & , & \quad \beta_0 = 0 , \\ \alpha_1 &= a_1 + a_2 & , & \quad \beta_1 = 1 - b_1 .\end{aligned}\tag{4.6}$$

On the other hand, for $q = 2$ we determine

$$\begin{aligned}\mathfrak{a}_0 &= -a_1 a_2 a_3 , \\ \mathfrak{a}_1 &= -(1 + a_1 + a_2 + a_3 + a_1 a_2 + a_1 a_3 + a_2 a_3) , \\ \mathfrak{a}_2 &= -(3 + a_1 + a_2 + a_3) , \\ \mathfrak{b}_1 &= -b_1 b_2 , \\ \mathfrak{b}_2 &= -(1 + b_1 + b_2) ,\end{aligned}\tag{4.7}$$

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{a_1 a_2 a_3}{1-x} & \frac{b_1 + b_2 + a_1 a_2 + a_1 a_3 + a_2 a_3 - b_1 b_2 - 1}{1-x} - \frac{1 - b_1 - b_2 + b_1 b_2}{x} & \frac{2 - b_1 - b_2}{x} - \frac{2 + a_1 + a_2 + a_3 - b_1 - b_2}{1-x} \end{pmatrix} ,$$

and with $T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & x^{-1} & 0 \\ 0 & -x^{-2} & x^{-2} \end{pmatrix}$ we arrive at:

$$B_0 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -\beta_1 & \beta_2 \end{pmatrix} , \quad B_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \alpha_0 & \alpha_1 - \beta_1 & \alpha_2 + \beta_2 \end{pmatrix} ,\tag{4.8}$$

with α_i being the $3-i$ -th elementary symmetric function of a_1, a_2, a_3 and β_i the $3-i$ -th elementary symmetric function of $1 - b_1, 1 - b_2$, i.e.

$$\begin{aligned}\alpha_0 &= a_1 a_2 a_3 & , & \quad \beta_0 = 0 , \\ \alpha_1 &= a_1 a_2 + a_2 a_3 + a_1 a_3 & , & \quad \beta_1 = (1 - b_1)(1 - b_2) , \\ \alpha_2 &= a_1 + a_2 + a_3 & , & \quad \beta_2 = 2 - b_1 - b_2 .\end{aligned}\tag{4.9}$$

For the generic case $n = q + 1$ one can recursively define the transformation (4.4)

$$T_n = \left(\begin{array}{c|c} T_{n-1} & \mathbf{0}_{n-1} \\ \hline x^{-1} \omega_1 \ T_{n-1} \ \omega_2 & x^{-n} \end{array} \right) ,\tag{4.10}$$

with

$$\omega_1 = (0^{n-2}, 1) \quad , \quad \omega_2 = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & -(n-2)! & 1 & 0 & \dots & 0 & 0 \\ \vdots & & & \ddots & \ddots & & \vdots \\ 0 & \dots & 0 & 1 & & 0 & 0 \\ 0 & \dots & 0 & -(n-2)! & & 1 & 0 \\ 0 & \dots & 0 & 0 & & 0 & -(n-2)! \end{pmatrix} ,$$

e.g.:

$$T_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{x} & 0 & 0 \\ 0 & -\frac{1}{x^2} & \frac{1}{x^2} & 0 \\ 0 & \frac{2}{x^3} & -\frac{3}{x^3} & \frac{1}{x^3} \end{pmatrix}, \quad T_5 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{x} & 0 & 0 & 0 \\ 0 & -\frac{1}{x^2} & \frac{1}{x^2} & 0 & 0 \\ 0 & \frac{2}{x^3} & -\frac{3}{x^3} & \frac{1}{x^3} & 0 \\ 0 & -\frac{6}{x^4} & \frac{11}{x^4} & -\frac{6}{x^4} & \frac{1}{x^4} \end{pmatrix}.$$

For (4.5) this transformation T yields the matrices

$$B_0 = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & (-1)^n \beta_1 & (-1)^{n+1} \beta_2 & \dots & -\beta_{n-2} & \beta_{n-1} \end{pmatrix}, \quad (4.11)$$

and

$$B_1 = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 \\ \alpha_0 & \alpha_1 + (-1)^n \beta_1 & \alpha_2 - (-1)^n \beta_2 & \dots & \alpha_{n-2} - \beta_{n-2} & \alpha_{n-1} + \beta_{n-1} \end{pmatrix}, \quad (4.12)$$

with α_i being the $n-i$ -th elementary symmetric function of a_1, \dots, a_n and β_i the $n-i$ -th elementary symmetric function of $1-b_1, \dots, 1-b_{n-1}$.

There is a whole family of transformations T yielding the form (4.5). E.g. for $q = 1$ the two transformations

$$T = \begin{pmatrix} 1 & 0 \\ 0 & \frac{\lambda}{1-x} \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 0 \\ 0 & \frac{\lambda}{x} \end{pmatrix}, \quad \lambda \in \mathbf{R} - \{0\} \quad (4.13)$$

yield

$$\begin{aligned} B_0 &= \begin{pmatrix} 0 & 0 \\ \frac{a_1 a_2}{\lambda} & -b_1 \end{pmatrix}, & B_1 &= \begin{pmatrix} 0 & \lambda \\ 0 & a_1 + a_2 - b_1 \end{pmatrix}, \\ B_0 &= \begin{pmatrix} 0 & \lambda \\ 0 & -b_1 \end{pmatrix}, & B_1 &= \begin{pmatrix} 0 & 0 \\ \frac{a_1 a_2}{c} & a_1 + a_2 - b_1 \end{pmatrix}, \end{aligned} \quad (4.14)$$

respectively.

The entries of the matrices (4.11) and (4.12) are given by the elementary symmetric functions α_r and β_s of the two sets of parameters a_1, \dots, a_n and $1-b_1, \dots, 1-b_{n-1}$, respectively. The latter naturally arise after expanding the differential equation (3.2), which yields:

$$-(1-x) \frac{d}{dx}(\theta^{n-1}y) + \sum_{i=1}^{n-1} \left(\alpha_i - \frac{\beta_i}{x} \right) \theta^i y + \alpha_0 y = 0, \quad \theta = x \frac{d}{dx}. \quad (4.15)$$

Furthermore, with

$$f_k = \theta^{(k-1)} y \quad , \quad k = 1, \dots, n \quad (4.16)$$

eq. (4.15) can be brought into the system (4.5) of first order linear differential equations with the matrices (4.11) and (4.12) and

$$\mathbf{g} = \begin{pmatrix} y \\ \theta y \\ \vdots \\ \theta^{(n-1)} y \end{pmatrix} \quad , \quad (4.17)$$

i.e. $\mathbf{f} = T\mathbf{g}$ with $T \equiv T_n$ given in (4.10).

For a specific initial condition the system (4.5) can be solved by Picard's iterative methods. A notorious example is the series expansion of the KZ equation to be discussed in the next subsection.

4.2 Drinfeld associator for expansions of hypergeometric functions

In this subsection we establish the connection between generalized hypergeometric functions ${}_{q+1}F_q$ and the Drinfeld associator. The KZ equation of one variable based on the free Lie algebra with generators e_0, e_1

$$\frac{d}{dx}\Phi = \left(\frac{e_0}{x} + \frac{e_1}{1-x} \right) \Phi \quad (4.18)$$

is the universal Fuchsian equation for the case of three regular singular points at 0, 1 and ∞ on \mathbf{P}^1 . The unique solution $\Phi(x) \in \mathbf{C}\langle\{e_0, e_1\}\rangle$ to (4.18) is known as the generating series of multiple polylogarithms (MPLs) in one variable [16]

$$\Phi(x) = \sum_{w \in \{e_0, e_1\}^*} L_w(x) w \quad , \quad (4.19)$$

with the coefficients $L_w(x)$ and the symbol $\{e_0, e_1\}^*$ denoting a non-commutative word $w = w_1 w_2 \dots$ in the letters $w_i \in \{e_0, e_1\}$. This alphabet specifies the underlying MPLs:

$$\begin{aligned} L_{e_0^n}(x) &:= \frac{1}{n!} \ln^n x \quad , \quad n \in \mathbf{N} \quad , \\ L_{e_1 w}(x) &:= \int_0^x \frac{dt}{1-t} L_w(t) \quad , \\ L_{e_0 w}(x) &:= \int_0^x \frac{dt}{t} L_w(t) \quad , \quad L_1(x) = 1 \quad . \end{aligned} \quad (4.20)$$

In particular, we have $L_{e_1} = -\ln(1-x)$ and $L_{e_0^{m-1}e_1}(x) = \mathcal{L}i_m(x)$, with the classical polylogarithm (3.12). The solution (4.19) can be found recursively and built by Picard's iterative methods. It is not possible to find power series solutions in x expanded at $x = 0$ or $x = 1$, because they have essential singularities at these points. However, one can construct a unique

analytic solution Φ_0 normalized at $x = 0$ with the asymptotic behaviour $\Phi_0(x) \rightarrow x^{e_0}$. Another fundamental solution Φ_1 normalized at $x = 1$ with $\Phi_1(x) \rightarrow (1-x)^{-e_1}$ can be considered. By analytic continuation the connection matrix between the solutions Φ_0 and Φ_1 is independent of x and gives rise to the Drinfeld associator:

$$Z(e_0, e_1) = \Phi_1(x)^{-1} \Phi_0(x) . \quad (4.21)$$

It is the regularized value of Φ at $x = 1$ and given by the non-commutative generating series of (shuffle-regularized) MZVs [17]

$$\begin{aligned} Z(e_0, e_1) = & \sum_{w \in \{e_0, e_1\}^*} \zeta(w) \quad w = 1 + \zeta_2 [e_0, e_1] + \zeta_3 ([e_0, [e_0, e_1]] - [e_1, [e_0, e_1]]) \\ & + \zeta_4 \left([e_0, [e_0, [e_0, e_1]]] - \frac{1}{4} [e_1, [e_0, [e_0, e_1]]] + [e_1, [e_1, [e_0, e_1]]] + \frac{5}{4} [e_0, e_1]^2 \right) \\ & + \zeta_2 \zeta_3 \left(([e_0, [e_0, e_1]] - [e_1, [e_0, e_1]]) [e_0, e_1] + [e_0, [e_1, [e_0, e_1]]] \right. \\ & \left. - [e_0, [e_1, [e_1, [e_0, e_1]]]] \right) + \zeta_5 \left([e_0, [e_0, [e_0, [e_0, e_1]]]] - \frac{1}{2} [e_0, [e_0, [e_1, [e_0, e_1]]]] \right. \\ & \left. - \frac{3}{2} [e_1, [e_0, [e_0, [e_0, e_1]]]] + (e_0 \leftrightarrow e_1) \right) + \dots , \end{aligned} \quad (4.22)$$

with $\zeta(e_0^{n_1-1} e_1 \dots e_0^{n_r-1} e_1) = \zeta(n_1, \dots, n_r)$, the shuffle relation $\zeta(w_1 \sqcup w_2) = \zeta(w_1) \zeta(w_2) = \zeta(w_1 w_2) + \zeta(w_2 w_1)$ and $\zeta(e_0) = 0 = \zeta(e_1)$.

By taking proper representations for e_0 and e_1 the KZ equation (4.18) gives rise to specific Fuchsian equations (4.5) describing generalized hypergeometric functions (3.1). Their local exponents at the regular singularities 0, 1 and ∞ are encoded in the Riemann scheme

$$\begin{pmatrix} \underline{0} & \underline{1} & \underline{\infty} \\ 0 & 0 & a_1 \\ 1-b_1 & 1 & a_2 \\ 1-b_2 & 2 & a_3 \\ \vdots & \vdots & \vdots \\ 1-b_{q-1} & q-1 & a_q \\ 1-b_q & d & a_{q+1} \end{pmatrix} , \quad (4.23)$$

with $d = -a_{q+1} + \sum_{j=1}^q b_j - a_j > 0$. The inequality condition guarantees convergence of (3.1) at the unit circle $|x| = 1$. For non-integer parameters b_j , with $b_i \neq b_j$ a $q+1$ -dimensional basis of solutions of (4.15) is given by [9]

$$\begin{aligned} & {}_{q+1}F_q \left[\begin{matrix} a_1, \dots, a_{q+1} \\ b_1, \dots, b_q \end{matrix}; x \right] , \\ & x^{1-b_i} {}_{q+1}F_q \left[\begin{matrix} 1+a_1-b_i, 1+a_2-b_i, \dots, 1+a_{q+1}-b_i \\ 1+b_1-b_i, 1+b_2-b_i, \dots, (*), \dots, 1+b_q-b_i, 2-b_i \end{matrix}; x \right] , \quad i = 1, \dots, q , \end{aligned} \quad (4.24)$$

with $(*)$ denoting omission of the expression $1+b_i-b_i$.

For (3.39) at $z = 1$ let us now derive an alternative expression, which is described by the Drinfeld associator. For this case the symmetric functions (3.43) can be used to parameterize the matrices (4.11) and (4.12) provided we change their form into:

$$B_0^p = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & -Q_{n-1}^p & -Q_{n-2}^p & \dots & -Q_2^p & -Q_1^p \end{pmatrix}, \quad B_1^p = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 \\ \Delta_n^p & \Delta_{n-1}^p & \dots & \Delta_2^p & \Delta_1^p \end{pmatrix}. \quad (4.25)$$

Now, Q_β^p is the β -th elementary symmetric function of b_1, \dots, b_{p-1} defined in (3.43) and Δ_β^p is given in (3.42), with $\beta = 1, \dots, n$ and $n = p$.

At a neighbourhood of $x = 0$ and $x = 1$ for (4.15) one can construct two sets of fundamental solutions u_i and v_i ($i = 1, \dots, n$), respectively. According to (4.17) for (4.5) these two sets give rise to the following solution matrices

$$\mathbf{g}_0 = \begin{pmatrix} u_1 & \dots & u_n \\ \theta u_1 & \dots & \theta u_n \\ \vdots & \vdots & \vdots \\ \theta^{(n-1)} u_1 & \dots & \theta^{(n-1)} u_n \end{pmatrix}, \quad \mathbf{g}_1 = \begin{pmatrix} v_1 & \dots & v_n \\ \theta v_1 & \dots & \theta v_n \\ \vdots & \vdots & \vdots \\ \theta^{(n-1)} v_1 & \dots & \theta^{(n-1)} v_n \end{pmatrix}, \quad (4.26)$$

respectively. The first set of fundamental solutions u_i is given in (4.24) subject to the change $b_l \rightarrow b_l + 1$. Hence, for $x \rightarrow 0$ we have:

$$\mathbf{g}_0 \xrightarrow{x \rightarrow 0} \begin{pmatrix} 1 & x^{-b_1} & \dots & x^{-b_q} \\ 0 & -b_1 x^{-b_1} & \dots & -b_q x^{-b_q} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & (-b_1)^{n-1} x^{-b_1} & \dots & (-b_q)^{n-1} x^{-b_q} \end{pmatrix} =: \gamma_0, \quad (4.27)$$

On the other hand, with (4.23) for $x \rightarrow 1$ we have

$$\mathbf{g}_1 \xrightarrow{x \rightarrow 1} \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ \theta v_1|_{x=1} & \theta v_2|_{x=1} & \dots & \theta v_{n-1}|_{x=1} & -d(1-x)^{d-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \theta^{(n-1)} v_1|_{x=1} & \theta^{(n-1)} v_2|_{x=1} & \dots & \theta^{(n-1)} v_{n-1}|_{x=1} & (-1)^{n-1} (d)_{n-1} (1-x)^{d-n+1} \end{pmatrix} =: \gamma_1, \quad (4.28)$$

with $(d)_{n-1}$ representing the falling factorial $(d)_{n-1} = d(d-1) \dots (d-n+2)$. We want to relate the solutions (4.26) to the normalized solutions Φ_i for the KZ equation (4.18) with e_0 and e_1 replaced by the representations B_0 and B_1 , respectively. By comparing the limits (4.27) and (4.28) with the behaviour of the normalized solutions Φ_i of (4.18)

$$\Phi_0 \xrightarrow{x \rightarrow 0} x^{B_0}, \quad \Phi_1 \xrightarrow{x \rightarrow 1} (1-x)^{-B_1} \quad (4.29)$$

one can define the following connection matrices C_0, C_1

$$\gamma_0 =: x^{B_0} C_0, \quad \gamma_1 =: (1-x)^{-B_1} C_1, \quad (4.30)$$

which allow to express the solutions (4.26) in terms of the normalized solutions Φ_0 and Φ_1 , respectively:

$$\mathbf{g}_i = \Phi_i C_i \quad , \quad i = 0, 1 \quad . \quad (4.31)$$

On the other hand, from the definition of the Drinfeld associator (4.21) it follows

$$Z(B_0, B_1) = C_1 \mathbf{g}_1^{-1} \mathbf{g}_0 C_0^{-1} \quad , \quad (4.32)$$

which is valid for any x . By considering in (4.32) the limit $x \rightarrow 1$ we are able to extract a relation between the first matrix element of the Drinfeld associator and the value of u_1 at $x = 1$, i.e.

$${}_p F_{p-1} \left[\begin{matrix} a_1, \dots, a_p \\ 1 + b_1, \dots, 1 + b_{p-1} \end{matrix} ; 1 \right] = Z(B_0^p, B_1^p) \big|_{1,1} \quad , \quad (4.33)$$

with e_0, e_1 replaced by the representations B_0 and B_1 , respectively. The latter can be found in (4.25). Eq. (4.32) is one of the main results of this subsection. E.g. for $p = 2$ and $p = 3$ we obtain

$${}_2 F_1 \left[\begin{matrix} a, b \\ 1 + c \end{matrix} ; 1 \right] = Z \left[\begin{pmatrix} 0 & 1 \\ 0 & -c \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ ab & a + b - c \end{pmatrix} \right] \bigg|_{1,1} \quad , \quad (4.34)$$

$${}_3 F_2 \left[\begin{matrix} a_1, a_2, a_3 \\ 1 + b_1, 1 + b_2 \end{matrix} ; 1 \right] = Z \left[\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -b_1 b_2 & -b_1 - b_2 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ a_1 a_2 a_3 & a_1 a_2 + a_1 a_3 + a_2 a_3 - b_1 b_2 & a_1 + a_2 + a_3 - b_1 - b_2 \end{pmatrix} \right] \bigg|_{1,1} \quad ,$$

respectively. For what follows it is important, that the transformations T given in (4.10) have a block structure w.r.t. the first entry. In order to prove (4.32) we derive the following block-structure of the matrices involved

$$\gamma_0 = \left(\begin{array}{c|c} 1 & x^{-b_1} \dots x^{-b_q} \\ \hline 0 & \mathcal{O}(x^{-b_1}) \dots \mathcal{O}(x^{-b_q}) \\ \vdots & \vdots \\ 0 & \mathcal{O}(x^{-b_1}) \dots \mathcal{O}(x^{-b_q}) \end{array} \right), \quad x^{B_0} = \left(\begin{array}{c|c} 1 & * \dots * \\ \hline 0 & \\ \vdots & * \\ 0 & \end{array} \right), \quad \text{i.e.: } C_0 = \left(\begin{array}{c|c} 1 & * \dots * \\ \hline 0 & \\ \vdots & * \\ 0 & \end{array} \right),$$

with irrelevant contributions of the order x^{-b_i} in the second block of γ_0 , and:

$$\gamma_1 = \left(\begin{array}{c|c} 1 & 0 \dots 0 \\ \hline * & \\ \vdots & * \\ * & \end{array} \right), \quad (1-x)^{-B_1} = \left(\begin{array}{c|c} 1 & 0 \dots 0 \\ \hline 0 & \\ \vdots & \mathbf{1}_{n-2} \\ 0 & \\ * & * \dots * \end{array} \right), \quad \text{i.e.: } C_1 = \left(\begin{array}{c|c} 1 & 0 \dots 0 \\ \hline * & \\ \vdots & * \\ * & \end{array} \right).$$

Eventually, with

$$\mathbf{g}_0(x=1) = \left(\begin{array}{c|c} u_1(x=1) & * \dots * \\ \hline * & \\ \vdots & * \\ * & \end{array} \right), \quad \mathbf{g}_1(x=1) \equiv \gamma_1 = \left(\begin{array}{c|c} 1 & 0 \dots 0 \\ \hline * & \\ \vdots & * \\ * & \end{array} \right)$$

we are able to verify (4.32).

Eq. (4.31) yields a connection between the hypergeometric solutions (4.26) and the fundamental solution Φ_0 of the KZ equation (4.18), which for the first matrix element reads:

$$\mathbf{g}_0|_{1,1} = \Phi_0 C_0|_{1,1} = \Phi_0[B_0^p, B_1^p](x)|_{1,1} . \quad (4.35)$$

The last relation follows from the block-structure of the connection matrix C_0 given above. Furthermore, an explicit expression for the fundamental solution Φ_0 is given in [18]

$$\begin{aligned} \Phi_0[e_0, e_1](x) = & 1 + e_0 \ln x - e_1 \ln(1-x) + \frac{1}{2} \ln^2 x e_0^2 + \mathcal{L}i_2(x) e_0 e_1 \\ & - [\mathcal{L}i_2(x) + \ln x \ln(1-x)] e_1 e_0 + \frac{1}{2} \ln^2(1-x) e_1^2 + \frac{1}{6} \ln^3 x e_0^3 \\ & + \mathcal{L}i_3(x) e_0^2 e_1 - [2 \mathcal{L}i_3(x) - \ln x \mathcal{L}i_2(x)] e_0 e_1 e_0 + \mathcal{L}i_{2,1}(x) e_0 e_1^2 \\ & + \left[\mathcal{L}i_3(x) - \ln x \mathcal{L}i_2(x) - \frac{1}{2} \ln^2 x \ln(1-x) \right] e_1 e_0^2 + \mathcal{L}i_{1,2}(x) e_1 e_0 e_1 \\ & - \left[\mathcal{L}i_{1,2}(x) + \mathcal{L}i_{2,1}(x) - \frac{1}{2} \ln x \ln^2(1-x) \right] e_1^2 e_0 - \frac{1}{6} \ln^3(1-x) e_1^3 + \dots , \end{aligned} \quad (4.36)$$

with the polylogarithms (3.12). This series follows by successively evaluating the sum (4.19). For generic p the first matrix element of \mathbf{g}_0 introduced in (4.26) represents the generalized hypergeometric function (3.39). With (4.36) we are able to generalize the relation (4.33) to arbitrary positions x

$${}_p F_{p-1} \left[\begin{matrix} a_1, \dots, a_p \\ 1 + b_1, \dots, 1 + b_{p-1} \end{matrix} ; x \right] = \Phi_0[B_0^p, B_1^p](x)|_{1,1} , \quad (4.37)$$

with e_0, e_1 replaced by the representations B_0^p and B_1^p , respectively. The latter can be found in (4.25). As a consequence of their block-structure eventually only a few terms of (4.36) contribute in (4.37), cf. eq. (5.27) as an example.

The series (3.39) is described for $z = 1$ by the relation (4.33) and for generic z by (4.37). It is worth pointing out the compactness and simplicity of the results (4.33) and (4.37). For a given power α'^k in (3.39) the coefficients $w_k^p(1)$ are given by the terms of degree k of the Drinfeld associator (4.22) described by MZVs of weight k supplemented by matrix products of k matrices B_0^p and B_1^p given in (4.25). Furthermore, for a given power α'^k in (3.39) the coefficients $w_k^p(z)$ are given by the terms of degree k of the fundamental solution (4.36) of the KZ equation. As a consequence, computing higher orders in the expansion (3.39) is reduced to simple matrix multiplications.

The entries of the matrices (4.25) are homogeneous polynomials of a given degree. This property makes sure, that the matrix elements $Z(B_0^p, B_1^p)|_{1,1}$ in eq. (4.33) and $\Phi_0[B_0^p, B_1^p](x)|_{1,1}$ in eq. (4.37) are also homogeneous in agreement with the expansion coefficients w_k^p in (3.39). The degree of the polynomials Q_β^p and Δ_β^p entering in the matrices (4.25) is in one-to-one correspondence with the corresponding number of derivatives of the corresponding entry of the vector (4.17). This fact is crucial to perform iterative methods at each order in the parameter ϵ .

Besides, for $z = 1$ in the series (3.39) the powers of α' are always accompanied by MZVs with a fixed “degree of transcendentality” (maximal transcendental). The degree of transcendentality (transcendentality level [19]) of π is defined to be 1, while that of $\zeta(n)$ is n , and for multiple zeta values $\zeta(n_1, \dots, n_r)$ it is $\sum_{i=1}^r n_i$. The degree of transcendentality for a product is defined to be the sum of the degrees of each factor. General graphical criteria for maximal transcendentality of multiple Gaussian hypergeometric functions have been given in [20, 21]. Similar transcendentality properties hold for the generic case $z \neq 1$.

4.3 Generic first-order Fuchsian equations, iterations and recurrences

A generic system $\mathbf{g}' = A(x)\mathbf{g}$ of n equations of (first-order) Fuchsian class has the form

$$\frac{d\mathbf{g}}{dx} = \sum_{i=0}^l \frac{A_i}{x - x_i} \mathbf{g} , \quad (4.38)$$

with the $l + 1$ distinct points x_0, \dots, x_l and constant quadratic non-commutative matrices $A_i \in M(n)$. If $\sum_{i=0}^l A_i \neq 0$ the system of equations (4.38) has $l + 2$ regular singular points at $x = x_i$ and $x = \infty$ and is known as Schlesinger system.

At a regular singular point any solution can be expressed explicitly by the combination of elementary functions and power series convergent within a circle around the singular point. A solution to (4.38) taking values in $\mathbf{C}\langle A \rangle$ with the alphabet $A = \{A_0, \dots, A_l\}$ can be given as formal weighted sum over iterated integrals (with the weight given by the number of iterated integrations)

$$\mathbf{g}(x) = \sum_{w \in A^*} L_w(x) w , \quad (4.39)$$

generalizing the case (4.19) and leading to hyperlogarithms [22]. The latter are defined recursively from words w built from an alphabet $\{w_0, w_1, \dots\}$ (with $w_i \simeq A_i$) with $l + 1$ letters:

$$\begin{aligned} L_{w_0^n}(x) &:= \frac{1}{n!} \ln^n(x - x_0) , \quad n \in \mathbf{N} , \\ L_{w_i^n}(x) &:= \frac{1}{n!} \ln^n\left(\frac{x - x_i}{x_0 - x_i}\right) , \quad 1 \leq i \leq l , \\ L_{w_i w}(x) &:= \int_0^x \frac{dt}{t - x_i} L_w(t) , \quad L_1(x) = 1 . \end{aligned} \quad (4.40)$$

The alphabet A is directly related to the differential forms

$$\frac{dx}{x - x_0}, \dots, \frac{dx}{x - x_l}$$

appearing in (4.38). Generically the functions (4.20) and (4.40) may also be written as Goncharov polylogarithms [23]

$$L_{w_{\sigma_1} \dots w_{\sigma_l}}(x) = G(x_{\sigma_1}, \dots, x_{\sigma_l}; x) , \quad (4.41)$$

given by

$$G(a_1, \dots, a_n; z) = \int_0^z \frac{dt}{t - a_1} G(a_2, \dots, a_n; t) , \quad (4.42)$$

with $G(z) := G(; z) = 1$ except $G(; 0) = 0$ and $a_i, z \in \mathbf{C}$. Typically, for a given class of amplitudes one only needs a certain special subset of allowed indices a_i referring to a specific alphabet. E.g. for the evaluation of loop integrals arising in massless quantum field theories one has $a_i \in \{0, 1\}$. However, the inclusion of particle masses in loop integrals may give rise to $a_i \in \{0, 1, -1\}$. The objects (4.42) are related to the MPLs (cf. also eq. (3.12))

$$\mathcal{L}i_{n_1, \dots, n_l}(z_1, \dots, z_l) = \sum_{0 < k_1 < \dots < k_l} \frac{z_1^{k_1} \dots z_l^{k_l}}{k_1^{n_1} \dots k_l^{n_l}} \quad (4.43)$$

as follows:

$$\mathcal{L}i_{n_1, \dots, n_l}(z_1, \dots, z_l) = (-1)^l G \left(\underbrace{0, \dots, 0}_{n_1-1}, \frac{1}{z_1}, \dots, \underbrace{0, \dots, 0}_{n_l-1}, \frac{1}{z_1 z_2 \dots z_l}; 1 \right) . \quad (4.44)$$

Furthermore we have:

$$L_{w_0^{n_l-1} w_{\sigma_l} \dots w_0^{n_2-1} w_{\sigma_2} w_0^{n_1-1} w_{\sigma_1}}(x) = (-1)^l \mathcal{L}i_{n_l, \dots, n_1} \left(\frac{x - x_0}{x_{\sigma_l} - x_0}, \frac{x_{\sigma_l} - x_0}{x_{\sigma_{l-1}} - x_0}, \dots, \frac{x_{\sigma_3} - x_0}{x_{\sigma_2} - x_0}, \frac{x_{\sigma_2} - x_0}{x_{\sigma_1} - x_0} \right) . \quad (4.45)$$

For any $0 \leq i \leq l$ there exists a unique solution $\mathbf{g}_i(x)$ to (4.38) with the leading behaviour $x \rightarrow x_i$ of \mathbf{g}_i given by (cf. eq. (4.30))

$$\mathbf{g}_i(x) = P_i(x) (x - x_i)^{A_i} \mathbf{C}_i , \quad (4.46)$$

with the normalization vector \mathbf{C}_i and some holomorphic power series

$$P_i = 1 + \sum_{n \geq 1} P_{in} (x - x_i)^n , \quad (4.47)$$

with complex coefficients P_{in} . Based on the building blocks (4.40) and by applying Picard's iterative methods a solution at $x = x_i$ can be constructed in a similar way as (4.39). One can compare two solutions referring to the two points $x = x_0$ and $x = x_i$. The quotient of any two such solutions is a non-commutative series with constant coefficients known as regularized zeta series giving rise to an associator:

$$Z^{(x_i)}(A_0, \dots, A_l) , \quad i = 1, \dots, l . \quad (4.48)$$

The latter determines the monodromy of the hyperlogarithms [24]. We refer to [25] for an explicit treatment of the case $l = 2$.

In the following let us assume, that in (4.38) the matrices A_i have some polynomial dependence on ϵ with integer powers as:

$$A_i = \sum_{n=1}^{n_0} a_{in} \epsilon^n . \quad (4.49)$$

In this case in (4.46) the factor \mathbf{C}_i and P_i contain subleading contributions in ϵ . We are looking for a power series solution in ϵ :

$$\mathbf{g}(x) = \sum_{k \geq 0} \mathbf{u}_k(x) \epsilon^k . \quad (4.50)$$

Eventually, each order ϵ^k of the power series is supplemented by a \mathbf{Q} -linear combination of iterated integrals of weight k . After inserting the Ansatz (4.50) into (4.38) we obtain a recursive differential equation for the functions $\mathbf{u}_k(x)$, which can be integrated to the following recursive relation

$$\mathbf{u}_k(x) = \mathbf{u}_k(0) + \sum_{i=0}^l \sum_{n=1}^{\min\{n_0, k\}} a_{in} \int_0^x \frac{\mathbf{u}_{k-n}(t)}{t - x_i} dt , \quad (4.51)$$

which translates into the following operator equation:

$$\mathbf{u}_k(x) = \mathbf{u}_k(0) + \sum_{i=0}^l \sum_{n=1}^{\min\{n_0, k\}} a_{in} I(x_i) \mathbf{u}_{k-n}(x) . \quad (4.52)$$

Above, $\mathbf{u}_k(0)$ represents a possible inhomogeneity accounting for an integration constant, which is determined by the initial value problem. Evidently, we have⁵ $\mathbf{u}_0(x) = \mathbf{u}_0(0) = \text{const}$. We may find a general solution to (4.52) by considering the following recurrence relation

$$\mathbf{u}_k(x) = \mathbf{u}_k(0) + \sum_{n=1}^{\min\{n_0, k\}} c_n \mathbf{u}_{k-n}(x) , \quad (4.53)$$

with the coefficients:

$$c_n = \sum_{i=0}^l a_{in} I(x_i) . \quad (4.54)$$

For (4.53) we can directly apply our general solution for inhomogeneous recurrence relations (2.34) to obtain

$$\mathbf{u}_k(x) = \sum_{\alpha=0}^{n_0-1} \sum_{\substack{|\vec{j}_1|+2|\vec{j}_2|+\dots+n_0|\vec{j}_{n_0}| \\ = k - n_0 - \alpha}} \{ \dots \} \sum_{\beta=\alpha+1}^{n_0} \sum_{\gamma=0}^l a_{\gamma\beta} I(x_\gamma) \bar{\mathbf{u}}_{n_0-\beta-\alpha} \quad (4.55)$$

$$+ \sum_{\alpha=n_0}^k \sum_{\substack{|\vec{j}_1|+2|\vec{j}_2|+\dots+n_0|\vec{j}_{n_0}| \\ = k - \alpha}} \{ \dots \} \mathbf{u}_\alpha(0) , \quad (4.56)$$

⁵Already at $k = 1$ the equation (4.51) translates into the non-trivial recursion $\mathbf{u}_1(x) = \mathbf{u}_1(0) + \sum_{i=0}^l a_{i1} \int_0^x \frac{\mathbf{u}_0(t)}{t - x_i} dt$.

with the generalized operator product

$$\{\dots\} = \{(\hat{I}_{1,0})^{j_{1,0}}, \dots, (\hat{I}_{1,l})^{j_{1,l}}, \dots, (\hat{I}_{n_0,0})^{j_{n_0,0}}, \dots, (\hat{I}_{n_0,l})^{j_{n_0,l}}\} , \quad (4.57)$$

and:

$$\hat{I}_{i,j} := a_{ji} I(x_j) . \quad (4.58)$$

Eq. (4.55) is valid for $k \geq n_0$ and it uses the initial values $\mathbf{u}_k(x) = \bar{\mathbf{u}}_k$, $k = 0, 1, \dots, n_0 - 1$. The vectors \vec{j}_i , $i = 1, \dots, n_0$ consist of $l + 1$ non-negative integers: $\vec{j}_i = (j_{i,0}, \dots, j_{i,l})$. Each of these elements $j_{a,b}$ is an index of a corresponding argument $\hat{I}_{a,b}$ in the generalized operator product (4.57).

Since the matrices a_{ji} are non-commutative they cannot be factorized from the generalized operator product. Therefore if possible it is better to avoid matrix notations. If the matrices a_{ji} commute the generalized operators product (4.57) simplifies to

$$\{\dots\} = \left(\prod_{\substack{0 \leq \beta \leq l \\ 1 \leq \gamma \leq n_0}} (a_{\beta,\gamma})^{j_{\gamma,\beta}} \right) \binom{j_{1,0} + \dots + j_{n_0,0}}{j_{1,0}, \dots, j_{n_0,0}} \dots \binom{j_{1,l} + \dots + j_{n_0,l}}{j_{1,l}, \dots, j_{n_0,l}} \quad (4.59)$$

$$\times \{I(x_0)^{j_{1,0} + \dots + j_{n_0,0}}, \dots, I(x_l)^{j_{1,l} + \dots + j_{n_0,l}}\} , \quad (4.60)$$

where every integral operator $I(x_i)$ appears only once.

The system (4.38) generically appears after expressing Feynman integrals as first-order coupled systems of differential equations, cf. e.g. Ref. [26] for some recent report on differential equations for Feynman integrals. In this context alternatively we could consider higher order differential equations instead of coupled first order equations, treat equations that can be decoupled as inhomogeneities and use our recurrence methods from section 2 and 3.

5 Low-energy expansion of superstring amplitudes

The full α' -dependence of tree-level string amplitudes is encoded in generalized Euler integrals, which integrate to multiple Gaussian hypergeometric functions [6]. Extracting from the latter the power series expansion in α' is of both phenomenological [27] and mathematical interest [4]. The computation of α' -expansions of generalized Euler integrals, which leads to MPLs integrating to MZVs, has been initiated in [6, 28], while a more systematic way by making profit of the underlying algebra of MPLs has been presented in [29]. Further attempts can be found in [30, 13, 31]. However, obtaining in a fully systematic way a closed, compact and analytic expression for a given order in α' , which does not rely on its lower orders to be computed in advance and which is straightforwardly applicable, is desirable. We have found two methods, which exactly meet these requirements, one way by solving the underlying recurrence relations (presented in section 3) and the other way by matching a given α' -order in the power series expansion with the corresponding coefficient of some fundamental and universal solution of the KZ equation (presented in section 4).

In this section we want to apply these two new methods to the four-point [32] and five-point [33, 6, 28] superstring disk amplitudes. An elegant and unifying picture of both amplitudes in terms of super Yang–Mills building blocks and generic string form factors has been elaborated in [34, 20, 21]. The latter are given by generalized hypergeometric functions (3.1) with $p = 2$ for the four-point case and $p = 3$ for the five-point case, respectively. The all-order expressions (3.38), (4.34) and (4.37) of these functions will now be used to obtain all-order expansions for these superstring amplitudes.

5.1 Four-point superstring amplitude

The four-point amplitude is written in terms of the hypergeometric function

$$F(\alpha's, \alpha'u) = {}_2F_1 \left[\begin{matrix} -\alpha's, & \alpha'u \\ 1 + \alpha's \end{matrix}; 1 \right], \quad (5.1)$$

with the Mandelstam variables $s = (k_1 + k_2)^2$ and $u = (k_1 + k_4)^2$. According to (3.13) and (3.33) the α' -expansion takes the following form:

$$F(\alpha's, \alpha'u) = 1 - \sum_{k=2}^{\infty} (-\alpha')^k \sum_{\alpha=1}^{k-1} s^{k-\alpha} u^{\alpha} \zeta(\alpha + 1, \{1\}^{k-\alpha-1}). \quad (5.2)$$

By this result the string corrections to the four-point Yang–Mills amplitudes can easily be calculated for any order in α' . The well known duality–symmetry of $F(s, u)$ w.r.t. the exchange $s \leftrightarrow u$ is not automatically fulfilled in (5.2). Instead, this leads to the relation

$$\zeta(\alpha_1 + 1, \{1\}^{\alpha_2-1}) = \zeta(\alpha_2 + 1, \{1\}^{\alpha_1-1}), \quad \alpha_1, \alpha_2 \geq 1, \quad (5.3)$$

which is a special case of the well-known duality formula for MZVs. Not only the identity (5.3) but also the representation (5.2) for $F(\alpha's, \alpha'u)$ is already known. However, it is interesting to compare these results with those for the five-point amplitude in the next sections.

5.2 Five-point superstring amplitude

For five-point amplitudes the basis of generalized Euler integrals is two-dimensional. A possible choice are the two functions [34, 20]

$$F_1 = F(\alpha's_1, \alpha's_2) F(\alpha's_3, \alpha's_4) {}_3F_2 \left[\begin{matrix} \alpha's_1, & 1 + \alpha's_4, & -\alpha's_{24} \\ 1 + \alpha's_1 + \alpha's_2, & 1 + \alpha's_3 + \alpha's_4 \end{matrix}; 1 \right], \quad (5.4)$$

$$F_2 = \alpha'^2 s_{13}s_{24} \frac{F(\alpha's_1, \alpha's_2) F(\alpha's_3, \alpha's_4)}{(1 + \alpha's_1 + \alpha's_2)(1 + \alpha's_3 + \alpha's_4)} {}_3F_2 \left[\begin{matrix} 1 + \alpha's_1, & 1 + \alpha's_4, & 1 - \alpha's_{24} \\ 2 + \alpha's_1 + \alpha's_2, & 2 + \alpha's_3 + \alpha's_4 \end{matrix}; 1 \right], \quad (5.5)$$

which depend on the kinematic invariants $s_{ij} = (k_i + k_j)^2$ and $s_i = s_{i\ i+1}$. Both hypergeometric functions can be related to (3.7) and derivatives thereof by using eqs. (3.4):

$${}_3F_2 \left[\begin{matrix} \alpha'a_1, & 1 + \alpha'a_2, & \alpha'a_3 \\ 1 + \alpha'b_1, & 1 + \alpha'b_2 \end{matrix}; 1 \right] = \left(\frac{\theta}{\alpha'a_2} + 1 \right) {}_3F_2 \left[\begin{matrix} \alpha'a_1, & \alpha'a_2, & \alpha'a_3 \\ 1 + \alpha'b_1, & 1 + \alpha'b_2 \end{matrix}; z \right] \Big|_{z=1}, \quad (5.6)$$

$${}_3F_2 \left[\begin{matrix} 1+\alpha'a_1, 1+\alpha'a_2, 1+\alpha'a_3 \\ 2+\alpha'b_1, 2+\alpha'b_2 \end{matrix}; 1 \right] = \frac{(1+\alpha'b_1)(1+\alpha'b_2)}{a_1 a_2 a_3 (\alpha')^3 z} \theta {}_3F_2 \left[\begin{matrix} \alpha'a_1, \alpha'a_2, \alpha'a_3 \\ 1+\alpha'b_1, 1+\alpha'b_2 \end{matrix}; z \right] \Big|_{z=1}, \quad (5.7)$$

with

$$\begin{aligned} a_1 &= s_1, & a_2 &= s_4, & a_3 &= s_2 + s_3 - s_5, \\ b_1 &= s_1 + s_2, & b_2 &= s_3 + s_4. \end{aligned} \quad (5.8)$$

Eqs. (5.6) and (5.7) lead to:

$$\begin{aligned} F_1 &= F(\alpha's_1, \alpha's_2) F(\alpha's_3, \alpha's_4) {}_3F_2 \left[\begin{matrix} \alpha's_1, \alpha's_4, -\alpha's_{24} \\ 1+\alpha's_1+\alpha's_2, 1+\alpha's_3+\alpha's_4 \end{matrix}; 1 \right] \\ &\quad - \alpha'^2 s_1 s_{24} F(\alpha's_1, \alpha's_2) F(\alpha's_3, \alpha's_4) \frac{\theta}{(\alpha')^3 \Delta_3} {}_3F_2 \left[\begin{matrix} \alpha's_1, \alpha's_4, -\alpha's_{24} \\ 1+\alpha's_1+\alpha's_2, 1+\alpha's_3+\alpha's_4 \end{matrix}; z \right] \Big|_{z=1}, \end{aligned} \quad (5.9)$$

$$F_2 = \alpha'^2 s_{13} s_{24} F(\alpha's_1, \alpha's_2) F(\alpha's_3, \alpha's_4) \frac{\theta}{(\alpha')^3 \Delta_3} {}_3F_2 \left[\begin{matrix} \alpha's_1, \alpha's_4, -\alpha's_{24} \\ 1+\alpha's_1+\alpha's_2, 1+\alpha's_3+\alpha's_4 \end{matrix}; z \right] \Big|_{z=1}. \quad (5.10)$$

In these two expressions the α' -expansions of all factors are known. As a consequence we are able to derive the expansions of F_1 and F_2 . In the following this is accomplished in two different ways. While in subsection 5.2.1 in the representation of (3.38) the expressions for the parameters (3.36) are left as symbols, in subsection 5.2.2 we take the explicit values for the parameters (3.36) subject to (5.8) and use the corresponding elementary symmetric functions:

$$\begin{aligned} \Delta_1 &= -s_5, \\ \Delta_2 &= s_1 s_2 - s_2 s_3 + s_3 s_4 - s_4 s_5 - s_5 s_1, \\ \Delta_3 &= s_1 s_2 s_4 + s_1 s_3 s_4 - s_1 s_4 s_5, \\ Q_1 &= s_1 + s_2 + s_3 + s_4, \\ Q_2 &= s_1 s_3 + s_2 s_3 + s_1 s_4 + s_2 s_4. \end{aligned} \quad (5.11)$$

With the parameters (5.11) we will derive an explicit expression for F_2 . Although this representation is not as compact as the one in subsection 5.2.1, it is more suitable for further applications to be discussed in section 6.

5.2.1 Representation in terms of elementary symmetric functions Δ_α and Q_α

Applying the results of section 3 for (5.9) and (5.10) yields

$$\begin{aligned} F_1 &= \sum_{k=0}^{\infty} (\alpha')^k \sum_{\substack{k_1, k_2 \geq 0 \\ k_1 + k_2 \leq k}} u_{k_1}(s_1, s_2) u_{k_2}(s_3, s_4) v_{k-k_1-k_2}(1) \\ &\quad - \sum_{k=2}^{\infty} (\alpha')^k s_1 s_{24} \sum_{\substack{k_1, k_2 \geq 0 \\ k_1 + k_2 \leq k-2}} u_{k_1}(s_1, s_2) u_{k_2}(s_3, s_4) \frac{\theta}{\Delta_3} v_{k-k_1-k_2+1}(z) \Big|_{z=1}, \end{aligned} \quad (5.12)$$

$$F_2 = \sum_{k=2}^{\infty} (\alpha')^k s_{13} s_{24} \sum_{\substack{k_1, k_2 \geq 0 \\ k_1 + k_2 \leq k-2}} u_{k_1}(s_1, s_2) u_{k_2}(s_3, s_4) \left. \frac{\theta}{\Delta_3} v_{k-k_1-k_2+1}(z) \right|_{z=1}, \quad (5.13)$$

with $u_k(a, b)$ defined in accordance with $u_k(z)$ in (3.32)

$$u_k(a, b) = \begin{cases} 1 & \text{for } k = 0, \\ \sum_{\alpha=1}^{k-1} (-1)^{k+1} a^{k-\alpha} b^{\alpha} \zeta(\alpha + 1, \{1\}^{k-\alpha-1}) & \text{otherwise,} \end{cases} \quad (5.14)$$

and with

$$\left. \frac{\theta}{\Delta_3} v_k(z) \right|_{z=1} = \sum_{l_1+m_1+2(l_2+m_2)+3m_3=k-3} (-1)^{l_1+l_2} \Delta_1^{m_1} \Delta_2^{m_2} \Delta_3^{m_3} Q_1^{l_1} Q_2^{l_2} \times I(0) \{I(0)^{l_1}, I(0, 0)^{l_2}, I(1)^{m_1}, I(1, 0)^{m_2}, I(1, 0, 0)^{m_3}\} I(1) \Big|_{z=1}, \quad (5.15)$$

representing the expansion of the $p = 3$ hypergeometric functions, which follows easily from (3.38). Eqs. (5.12) and (5.13) give all orders of the α' -expansions of the five-point open superstring amplitude. Besides the kinematic variables at each order only products of at most three MZVs are produced. This is to be contrasted with the procedure in [29] where also higher products of MZVs appear.

As already discussed at the end of subsection 3.3 there is a way to remove the generalized operator product to obtain a representation in terms of MZVs.

5.2.2 Representation in terms of kinematic invariants s_i

The polynomial parts of $u_{k_1}(s_1, s_2)$, $u_{k_2}(s_3, s_4)$ and $v_k(z)$ can be combined in (5.13) to obtain

$$F_2 = s_{13} s_{24} \sum_{k=2}^{\infty} (\alpha')^k \sum_{j_1+j_2+j_3+j_4+j_5=k-2} s_1^{j_1} s_2^{j_2} s_3^{j_3} s_4^{j_4} s_5^{j_5} f(j_1, j_2, j_3, j_4, j_5), \quad (5.16)$$

where all MZVs and integral operators are contained in:

$$f(j_1, j_2, j_3, j_4, j_5) = \sum_{\vec{l}=(l_1, l_2, l_3, l_4)} (-1)^{|\vec{l}|} \zeta'_{l_1, l_2} \zeta'_{l_3, l_4} v(j_1 - l_1, j_2 - l_2, j_3 - l_3, j_4 - l_4, j_5). \quad (5.17)$$

The function (5.17) involves the MZVs of (5.14)

$$\zeta'_{i_1, i_2} = \begin{cases} \zeta(i_1 + 1, \{1\}^{i_2-1}) & \text{for } i_1, i_2 \geq 1, \\ -1 & \text{for } i_1, i_2 = 0, \\ 0 & \text{else,} \end{cases} \quad (5.18)$$

and the integral operators of (5.15):

$$v(j_1, j_2, j_3, j_4, j_5)$$

$$\begin{aligned}
&= \sum_{\vec{\alpha}, \vec{\beta}, \gamma, \vec{\delta}, \vec{\epsilon} \in L} I(0) \left\{ I(0)^{|\vec{\alpha}|}, I(0,0)^{|\vec{\beta}|}, I(1)^\gamma, I(1,0)^{|\vec{\delta}|}, I(1,0,0)^{|\vec{\epsilon}|} \right\} I(1) \Big|_{z=1} \\
&\times (-1)^{|\vec{\alpha}|+|\vec{\beta}|+\delta_3+j_5} \binom{|\vec{\alpha}|}{\alpha_1, \alpha_2, \alpha_3, \alpha_4} \binom{|\vec{\beta}|}{\beta_1, \beta_2, \beta_3, \beta_4} \binom{|\vec{\delta}|}{\delta_1, \delta_2, \delta_3, \delta_4, \delta_5} \binom{|\vec{\epsilon}|}{\epsilon_1, \epsilon_2, \epsilon_3} .
\end{aligned} \tag{5.19}$$

The summation is over non-negative integers γ and the multiple indices:

$$\begin{aligned}
\vec{\alpha} &= (\alpha_1, \alpha_2, \alpha_3, \alpha_4) , \\
\vec{\beta} &= (\beta_1, \beta_2, \beta_3, \beta_4) , \\
\vec{\delta} &= (\delta_1, \delta_2, \delta_3, \delta_4, \delta_5) , \\
\vec{\epsilon} &= (\epsilon_1, \epsilon_2, \epsilon_3) .
\end{aligned} \tag{5.20}$$

The summation region of $\vec{\alpha}$, $\vec{\beta}$, γ , $\vec{\delta}$ and $\vec{\epsilon}$ is the solution set L of the five equations:

$$\begin{aligned}
j_1 &= \alpha_1 + \beta_1 + \beta_2 + \delta_1 + \delta_2 + |\vec{\epsilon}| , \\
j_2 &= \alpha_2 + \beta_3 + \beta_4 + \delta_1 + \delta_3 + \epsilon_1 , \\
j_3 &= \alpha_3 + \beta_1 + \beta_3 + \delta_3 + \delta_4 + \epsilon_2 , \\
j_4 &= \alpha_4 + \beta_2 + \beta_4 + \delta_4 + \delta_5 + |\vec{\epsilon}| . \\
j_5 &= \gamma + \delta_2 + \delta_5 + \epsilon_3 .
\end{aligned} \tag{5.21}$$

The function $v(j_1, j_2, j_3, j_4, j_5)$ is related to $v_k(z)$ through:

$$\left. \frac{\theta}{\Delta_3} v_k(z) \right|_{z=1} = \sum_{j_1+j_2+j_3+j_4+j_5=k-3} s_1^{j_1} s_2^{j_2} s_3^{j_3} s_4^{j_4} s_5^{j_5} v(j_1, j_2, j_3, j_4, j_5) . \tag{5.22}$$

Obviously, the r.h.s. of (5.22) together with (5.19) is less compact than the r.h.s. of (5.15), which uses the quantities Δ_1 , Δ_2 , Δ_3 , Q_1 and Q_2 . But the advantage of the former is, that the symmetry of $F_2(s_{13}s_{24})^{-1}$ can directly be analyzed in (5.17). This function is invariant w.r.t. cyclic permutation of the kinematic invariants $(s_1, s_2, s_3, s_4, s_5)$. Therefore $f(j_1, j_2, j_3, j_4, j_5)$ is invariant w.r.t. cyclic permutations of $(j_1, j_2, j_3, j_4, j_5)$. Just like for the four-point amplitude (cf. subsec. 5.1) this symmetry is non-trivially fulfilled and various MZV identities are generated. To obtain them, the generalized operator product in (5.19) has to be written in terms of MZVs. This is of course the same operator product as the one in (3.38) and (5.15). In addition, there are summations over a total of 17 indices, of which five can be evaluated with eqs. (5.21). Thus besides the issue of converting the generalized product of integral operators into MZVs it is interesting to see, whether some of the remaining twelve sums can be evaluated. For the α' -expansion of the 4-point amplitude (5.2), this is performed in section 3.2. The identity, which leads from eq. (3.31) to (3.32) eliminates both the generalized operator product and the inner sum. This identity and similar ones, which apply to the 5-point case, are discussed in section 6.

5.3 Open superstring amplitudes and the Drinfeld associator

World-sheet disk integrals describing N -point open string tree-level amplitudes integrate to multiple Gaussian hypergeometric functions [6] whose singularity structure is generically given

by divisors whose monodromy is described by generalized KZ equations. Prime examples are the cases of $N = 4$ and $N = 5$ whose singularity structure is described by the three regular singular points $0, 1$ and ∞ leading to the Fuchsian differential equations of first order (4.5). The latter is related to the KZ equation (4.18), whose basic solutions can be adjusted to specific solutions of (4.5), cf. section 4 for more details. Hence, the solutions of the KZ equation are *directly* related to the world-sheet disk integrals and their singularity structure. As a consequence the monodromies on the string world-sheet are *explicitly* furnished by the Drinfeld associator. By contrast in [30] a relation between the Drinfeld associator (4.22) and N -point world-sheet disk integrals has been given by connecting the boundary values of the N -point and $N - 1$ -point integrals thereby giving a relation between their underlying world-sheet disk integral.

The four-point superstring amplitude is written in terms of the hypergeometric function (5.1). With the choice

$$a \equiv a_1 = -\alpha' s \quad , \quad b \equiv a_2 = \alpha' u \quad , \quad c \equiv b_1 = \alpha' s \quad (5.23)$$

the matrices (4.25) become:

$$B_0 = \begin{pmatrix} 0 & 1 \\ 0 & -\alpha' s \end{pmatrix} \quad , \quad B_1 = \begin{pmatrix} 0 & 0 \\ -(\alpha')^2 su & \alpha'(u - 2s) \end{pmatrix} . \quad (5.24)$$

With this matrix representation eq. (4.34) expresses the hypergeometric function (5.1) in terms of the Drinfeld associator Z as:

$${}_2F_1 \left[\begin{matrix} -\alpha' s, \alpha' u \\ 1 + \alpha' s \end{matrix} ; 1 \right] = Z(B_0, B_1)|_{1,1} . \quad (5.25)$$

This gives a *direct* relation between the four-point superstring amplitude and the Drinfeld associator by *exactly* matching the monodromy of the string world-sheet to the corresponding monodromy of the underlying hypergeometric function ${}_2F_1$. We refer the reader to [35] for a different relation with a matrix representation which is linear in the kinematic invariants.

The five-point superstring amplitude uses the two functions (5.4) and (5.5). According to (5.6) and (5.7) the latter can be generated from the single function

$${}_3F_2 \left[\begin{matrix} \alpha' a_1, \alpha' a_2, \alpha' a_3 \\ 1 + \alpha' b_1, 1 + \alpha' b_2 \end{matrix} ; x \right] , \quad (5.26)$$

with the parameters a_i and b_j defined in (5.8). According to the relation (4.37) the function (5.26) can be related to the fundamental solution Φ_0 of the KZ equation (4.18). For $p = 3$ we obtain

$$\begin{aligned} {}_3F_2 \left[\begin{matrix} \alpha' a_1, \alpha' a_2, \alpha' a_3 \\ 1 + \alpha' b_1, 1 + \alpha' b_2 \end{matrix} ; x \right] &= \Phi_0[B_0, B_1](x)|_{1,1} \\ &= 1 + \Delta_3 \mathcal{L}i_3(x) + \Delta_1 \Delta_3 \mathcal{L}i_{2,2}(x) - Q_1 \Delta_3 \mathcal{L}i_4(x) + \dots , \end{aligned} \quad (5.27)$$

with the matrix representations (4.25)

$$B_0 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -\alpha'^2 Q_2 & -\alpha' Q_1 \end{pmatrix} \quad , \quad B_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \alpha'^3 \Delta_3 & \alpha'^2 \Delta_2 & \alpha' \Delta_1 \end{pmatrix} , \quad (5.28)$$

and the parameters Q_α and Δ_β corresponding to the elementary symmetric functions and given in (5.11). Hence, all details for the two functions (5.4) and (5.5) can be derived from the fundamental solution (5.27) of the KZ equation (4.5). To summarize, the whole α' -dependence of the five-point superstring amplitude is described by the KZ equation (4.5) and its fundamental solution (4.36) with the representations (5.28). Again, this gives a *direct* relation between the five-point superstring amplitude and the Drinfeld associator by *exactly* matching the monodromy of the string world-sheet to the corresponding monodromy of the underlying hypergeometric function ${}_3F_2$.

Similar relations can be found for any N -point superstring amplitude (with $N \geq 6$). This will be exhibited in a future publication [36].

6 From generalized operator products to MZVs

The results of the sections 3 and 5 left some open questions: how to obtain the all-order expression (3.32) for the hypergeometric function ${}_2F_1$ from its representation (3.31) with the latter involving generalized operator products. Likewise, how to achieve similar transformations on the operator products arising for the hypergeometric functions ${}_3F_2$ and ${}_pF_{p-1}$ in general. These questions will be discussed in section 6.1. The relations to be derived there will be applied to the results from sections 3 and 5 in section 6.2. The MZV identities, which follow from cyclic symmetry of the function $f(j_1, j_2, j_3, j_4, j_5)$ will be discussed in section 6.3.

6.1 Identities for generalized operator products

In this subsection three types of operator products will be discussed. Starting from simple cases involving independent arguments the complexity increases step by step to finally obtain identities, which can be applied to (3.38), (3.45) and (5.19). Therefore, not all identities will be needed, at least not for the results of this paper. However, simpler identities provide a consistency check for the most complicated ones.

All identities for generalized operator products presented in this section contain MZVs. Identical relations hold for MPLs as well. Before and after every generalized operator product there is an $I(0)$ and an $I(1)$ operator, respectively, to ensure finiteness of the corresponding MZV. The following equations make extensive use of two notations, which we introduced at the end of section 3.1 in eqs. (3.20) - (3.26): sums over all sets of indices \vec{n} of MZVs $\zeta(\vec{n})$ and multiple index sums.

6.1.1 Independent arguments

We start with generalized operator products, which include only independent arguments.

Example 1.1: The simplest example is:

$$I(0)\{I(0)^{j_1}, I(1)^{j_2}\}I(1)\big|_{z=1} = \sum_{\substack{w=j_1+j_2+2 \\ d=j_2+1}} \zeta(\vec{n}) . \quad (6.1)$$

It is clear, that the sum is over the given weight and depth, since these quantities correlate directly to the number of integral operators. It needs to be proven, that *all* MZVs of given weight and depth are generated by the generalized operator product on the l.h.s. To do this, it is sufficient to show, that

1. both sides contain the same number of terms
2. and that there are no identical terms on the l.h.s.

The second point is clear due to the definition of the generalized operator product as the sum of *distinct* permutations and the fact, that both arguments $I(0)$ and $I(1)$ are independent. The first property is also true. The number of different MZVs of weight w and depth d is $\binom{w-2}{d-1}$, which equals $\binom{j_1+j_2}{j_2}$ in this case. According to eq. (2.9) this quantity is identical to the number of terms on the l.h.s. of (6.1).

Example 1.2: With $I(1,0)$ instead of $I(1)$ the identity (6.1) becomes:

$$I(0)\{I(0)^{j_1}, I(1,0)^{j_2}\}I(1)\big|_{z=1} = \sum_{\substack{w=j_1+2j_2+2 \\ d=j_2+1; \ n_i \geq 2}} \zeta(\vec{n}) . \quad (6.2)$$

The additional condition $n_i \geq 2$ is self-explanatory. For the first integer n_1 it is obvious that $n_1 \geq 2$, because the l.h.s. starts with the operator $I(0)$. For any other integers n_i to be one, any sequence of integral operators would have to include the product $I(1,1)$. With the arguments $I(0)$ and $I(1,0)$ this is obviously not possible. To prove that the l.h.s. of (6.2) produces *all* MZVs of given weight and depth, which do not include an index $n_i = 1$, similar arguments as for (6.1) hold here. The generalization to cases with $I(1,0)$ replaced by other arguments of the type $I(1,0, \dots, 0)$ is straightforward.

Examples 1.3 and 1.4: Other generalized operator products with independent arguments are

$$\begin{aligned} I(0)\{I(1)^{j_1}, I(1,0)^{j_2}\}I(1)\big|_{z=1} &= I(0,1)\{I(1)^{j_1}, I(0,1)^{j_2}\}\big|_{z=1} \\ &= \sum_{\substack{w=j_1+2j_2+2; \ d=j_1+j_2+1 \\ n_1=2; \ d_1=j_1; \ d_2=j_2+1}} \zeta(\vec{n}) , \end{aligned} \quad (6.3)$$

and

$$I(0)\{I(1)^{j_1}, I(1,0)^{j_2}, I(1,0,0)^{j_3}\}I(1)\big|_{z=1} = I(0,1)\{I(1)^{j_1}, I(0,1)^{j_2}, I(0,0,1)^{j_3}\}\big|_{z=1}$$

$$= \sum_{\substack{w=j_1+2j_2+3j_3+2; \ d=j_1+j_2+j_3+1 \\ n_1=2; \ d_1=j_1; \ d_2=j_2+1; \ d_3=j_3}} \zeta(\vec{n}) , \quad (6.4)$$

with two and three arguments, respectively. In both first equations we used

$$\{I(1)^{j_1}, I(1, 0)^{j_2}, \dots, I(1, \underbrace{0, \dots, 0}_{j_n-1})^{j_n}\} I(1) = I(1) \{I(1)^{j_1}, I(0, 1)^{j_2}, \dots, I(\underbrace{0, \dots, 0}_{j_n-1}, 1)^{j_n}\} , \quad (6.5)$$

to make evident the conditions on d_i . To check consistency, note that $d = \sum_i d_i$ and that relation (6.4) becomes (6.3) for $j_3 = 0$. For a strict proof, the same strategy as for (6.1) should work. Even though more combinatorics is needed here, to determine for example the number of MZVs of given w , d , d_1 , d_2 and d_3 . The generalization to more arguments of the kind $I(1, 0, \dots, 0)$ is straightforward.

6.1.2 Dependent arguments

The following identities involve generalized products of dependent operators. Distinct permutations of dependent factors can be identical. Consider for instance the generalized operator product

$$\{I(0), I(1), I(1, 0)\} = I(0, 1, 1, 0) + I(0, 1, 0, 1) + I(1, 0, 0, 1) + I(1, 1, 0, 0) + 2I(1, 0, 1, 0) . \quad (6.6)$$

Clearly, the third arguments $I(1, 0)$ can be written as a product of the first two $I(1)$ and $I(0)$. As a consequence the two distinct permutations

$$I(1)I(0)I(1, 0) \text{ and } I(1, 0)I(1)I(0) \quad (6.7)$$

are identical and the corresponding product $I(1, 0, 1, 0)$ appears twice in (6.6). No other products appears more than once, since $I(1, 0, 1, 0)$ is the only one to contain twice the sequence $I(1)I(0)$. For the more complicated identities in the following, the task is to count identical permutations such as (6.7). Since these identities translate generalized operator products into sums of MZVs, identical permutations correspond to MZVs, which appear more than once. Therefore a weighting is required in the sum of MZVs, i.e. a function, which can be evaluated for every single MZV in that sum and thereby describes how often every single MZV appears. On one hand weightings can depend on quantities, which are identical for all MZVs of the corresponding sum. These quantities are determined by the relevant generalized operator product. Examples are the indices of the generalized operator product or the weight w and depth d , which are given by the number of integral operators. On the other hand, in order to give different factors for different MZVs, the weighting has to depend on the indices $\vec{n} = (n_1, \dots, n_d)$ of MZVs $\zeta(\vec{n})$. This dependence can be explicit or in terms of related quantities such as the number d_j of indices, which equal j .

We start with the discussion on the generalized operator product of the most general case (3.45). There are two types of arguments in the generalized operator products we encountered in our results of sections 3 and 5. Those, that consist only of integral operators $I(0)$ and those,

which have an operator $I(1)$ to the left of all $I(0)$ operators. In the following relation the former have an index j_μ , $\mu = 1, \dots, a$ and the latter have j'_ν , $\nu = 1, \dots, b$:

$$\begin{aligned} I(0)\{I(0)^{j_1}, I(0,0)^{j_2}, \dots, I(\underbrace{0, \dots, 0}_a)^{j_a}, I(1)^{j'_1}, I(1,0)^{j'_2}, \dots, I(1, \underbrace{0, \dots, 0}_{b-1})^{j'_b}\}I(1)|_{z=1} \\ = \sum_{\substack{w=j_1+2j_2+\dots+aj_a \\ +j'_1+2j'_2+\dots+bj'_b+2 \\ d=j'_1+j'_2+\dots+j'_b+1}} \zeta(\vec{n}) \omega_{a,b}(\vec{n}' - \vec{1}; j_1, j_2, \dots, j_a; j'_1, j'_2, \dots, j'_b) , \end{aligned} \quad (6.8)$$

with the constant vector $\vec{1} = (\underbrace{1, \dots, 1}_d)$ and the elements

$$n'_i = \begin{cases} n_1 - 1 & \text{for } i = 1 , \\ n_i & \text{for } i = 2, \dots, d , \end{cases} \quad (6.9)$$

of the vector \vec{n}' . The conditions for the weight w and the depth d in the sum of MZVs are self-explanatory. They are related to the number of integral operators, which can easily be read off from the first line. Of greater interest is the weighting

$$\begin{aligned} \omega_{a,b}(\vec{n}; j_1, j_2, \dots, j_a; j'_1, j'_2, \dots, j'_b) \\ = \sum_{\substack{\vec{\beta}_1 + \vec{\beta}_2 + \dots + \vec{\beta}_b = \vec{1} \\ |\vec{\beta}_\nu| = j'_\nu; \beta_{\nu,1} = 0}} \omega_a(\vec{n} - \vec{\beta}_2 - 2\vec{\beta}_3 - \dots - (b-1)\vec{\beta}_b; j_1, j_2, \dots, j_a) , \end{aligned} \quad (6.10)$$

with

$$\omega_a(\vec{n}; j_1, j_2, \dots, j_a) = \sum_{\substack{\vec{\alpha}_1 + 2\vec{\alpha}_2 + \dots + a\vec{\alpha}_a = \vec{n} \\ |\vec{\alpha}_\mu| = j_\mu}} \begin{pmatrix} \vec{\alpha}_1 + \vec{\alpha}_2 + \dots + \vec{\alpha}_a \\ \vec{\alpha}_1, \vec{\alpha}_2, \dots, \vec{\alpha}_a \end{pmatrix} . \quad (6.11)$$

The weighting $\omega_{a,b}$ depends on the indices \vec{n} of the MZVs and the indices j_μ and j'_ν of the generalized operator product. Besides the multinomial coefficient, essentially there are two summations in the definitions (6.10) and (6.11). The one in (6.10) over the indices

$$\vec{\beta}_\nu = (\beta_{\nu,1}, \dots, \beta_{\nu,d}) , \quad \nu = 1, \dots, b \quad (6.12)$$

take permutations into account, which involve the operators $I(1,0,\dots,0)$, while the sums in (6.11) over the indices

$$\vec{\alpha}_\mu = (\alpha_{\mu,1}, \dots, \alpha_{\mu,d}) , \quad \mu = 1, \dots, a \quad (6.13)$$

refer to the operators of the type $I(0,\dots,0)$. To explain the expressions (6.10) and (6.11) in detail, we consider permutations of the arguments of the type $I(0,\dots,0)$ before discussing those of the arguments of the type $I(1,0,\dots,0)$. Finally we explain how both types are related.

Both sums in (6.10) and (6.11) use several multi-indices (6.12) and (6.13) with d elements each. This is motivated by the following idea. First we count the number of identical permutations of operators, which are part of the integral operator representation

$$I(\underbrace{0, \dots, 0}_{n_i-1}, 1) \quad (6.14)$$

of a single MZV index n_i . Then we use the multi-index notation to combine the results of all indices \vec{n} , which yields the total number of identical permutations, i.e. the weighting.

Let us start with identical permutations of arguments of the type $I(0, \dots, 0)$, which are part of one index (6.14). Assuming there are a different arguments of the type $I(\underbrace{0, \dots, 0}_{\mu}) \equiv I(0)^\mu$, $\mu = 1, \dots, a$, which appear α_μ times, then for fixed α_μ there are

$$\binom{\alpha_1 + \alpha_2 + \dots + \alpha_a}{\alpha_1, \alpha_2, \dots, \alpha_a} \quad (6.15)$$

distinct permutations of these arguments. Every permutation is a sequence of $\alpha_1 + 2\alpha_2 + \dots + a\alpha_a$ operators $I(0)$. Now we assume the α_μ are not fixed, but the total number of operators $I(0)$ has to add up to $n_i - 1$. To count the permutations we have to sum over all sets $(\alpha_1, \alpha_2, \dots, \alpha_a)$ and take the fixed total number of $n_i - 1$ operators $I(0)$ into account:

$$\sum_{\alpha_1 + 2\alpha_2 + \dots + a\alpha_a = n_i - 1} \binom{\alpha_1 + \alpha_2 + \dots + \alpha_a}{\alpha_1, \alpha_2, \dots, \alpha_a} . \quad (6.16)$$

These are the number the possibilities to permute the arguments of the type $I(0, \dots, 0)$, which contribute to one index (6.14). Multiplying the possibilities of all indices \vec{n} of a MZV yields the multi-index notation presented in (6.11) and (6.13), with $\alpha_{\mu,i}$ being the number of operators $I(0)^\mu$, which contribute to the index n_i . The elements of a multi-index are not independent, since their sum $\alpha_{\mu,1} + \dots + \alpha_{\mu,d} = |\vec{\alpha}_\mu|$, i.e. the total number of operators $I(0)^\mu$, is fixed by the index j_μ of the generalized operator product. As a consequence we have the additional conditions $|\vec{\alpha}_\mu| = j_\mu$, $\mu = 1, \dots, a$ in (6.11). The first index n_1 is an exceptional case. The first $I(0)$ in the integral operator representation is the one to the left of the generalized operator product in (6.8). Thus only $n_1 - 2$ operators are relevant for the permutations. This explains the use of (6.9) for the arguments of the weighting $\omega_{a,b}$ in the identity (6.8).

Now that we have discussed the arrangements of the arguments $I(0, \dots, 0)$, described by ω_a , let us consider the arguments $I(1, 0, \dots, 0)$. All identical permutations of the b arguments $I(1, \underbrace{0, \dots, 0}_{\nu-1}) \equiv I(1)I(0)^{\nu-1}$, $\nu = 1, \dots, b$ are taken into account by the multi-index sums in

(6.10). The summation indices (6.12) take only two different values:

$$\beta_{\nu,i} \in \{0, 1\} , \quad \nu = 1, \dots, b , \quad i = 2, \dots, d . \quad (6.17)$$

The case $\beta_{\nu,i} = 1$ corresponds to $I(1)I(0)^{\nu-1}$ contributing⁶ to the index n_{i-1} . On the other hand $\beta_{\nu,i} = 0$ means, that $I(1)I(0)^{\nu-1}$ does not contribute to n_{i-1} . Of course no more (and no

⁶By saying “ n_i originates from c_j ” or “ c_j contributes to n_i ” it is meant, that $I(1)$ in the integral operator representation (6.14) of the index n_i is part of the operator c_j .

less) than one argument of the type $I(1, 0, \dots, 0)$ can contribute to a single index n_i , therefore (6.10) uses the conditions $\beta_{1,i} + \dots + \beta_{b,i} = 1$ for all $i = 1, \dots, d$ or in the multi-index notation: $\vec{\beta}_1 + \dots + \vec{\beta}_b = \vec{1}$. Similar to the conditions for $|\vec{\alpha}_\mu|$ in (6.11), the conditions for $|\vec{\beta}_\nu|$ are necessary due to the fixed total number j'_ν of arguments $I(1)I(0)^{\nu-1}$ in the generalized operator product. The first index n_1 provides again an exceptional case. Since there is no previous index to which any argument could contribute, we set $\beta_{\nu,1} = 0$, $\nu = 1, \dots, b$.

Finally we can analyse how permutations between arguments of the type $I(0, \dots, 0)$ and those between arguments $I(1, 0, \dots, 0)$ affect each other. For every configuration of the arguments of the type $I(1, 0, \dots, 0)$, i.e. for every set of indices $\beta_{\nu,i}$, we need to count the identical permutations of the arguments of the type $I(0, \dots, 0)$ via ω_a . That is why ω_a appears in (6.10). However, this combination of the two types of permutations is not just a product, since they are not independent. In other words, it is not possible to obtain identical products, out of permutations of two operators $I(1)I(0)^{\nu_1}$ and $I(1)I(0)^{\nu_2}$ with $\nu_1 \neq \nu_2$, without changing the positions of operators of the type $I(0, \dots, 0)$ as well. This effect of the sums over (6.12) on the sums over (6.13) is described by the first argument of ω_a in (6.10). It contains the contributions $-\vec{\beta}_2 - 2\vec{\beta}_3 - \dots - (b-1)\vec{\beta}_b$. As a result, after inserting (6.10) and (6.11) in (6.8), we get the conditions

$$\alpha_{1,i} + 2\alpha_{2,i} + \dots + a\alpha_{a,i} = n'_i - 1 - \beta_{2,i} - 2\beta_{3,i} - \dots - (b-1)\beta_{b,i}, \quad i = 1, \dots, d \quad (6.18)$$

for the sums over the indices (6.13). This can be explained as follows. Recall, that the l.h.s. represents the number of operators $I(0)$, which are relevant for permutations of arguments $I(0, \dots, 0)$. In general this does not equal $n'_i - 1$ as suggested by (6.16) and the discussions in that paragraph. It depends on which of the arguments $I(1, 0, \dots, 0)$ contributes to the previous index n_{i-1} . E.g. there are only $n_i - 2$ relevant operators in case $I(1, 0)$ contributes to n_{i-1} ($i > 1$), since the first $I(0)$ in (6.14) comes from the argument $I(1, 0)$ and is therefore fixed. This case is represented by $\beta_{2,i} = 1$, which indeed gives $n_i - 2$ on the r.h.s. of (6.18). In general the number of relevant operators $I(0)$ is $n_i - \nu$ with $I(1)I(0)^{\nu-1}$ contributing to n_{i-1} . In accordance with eq. (6.18) this is represented by $\beta_{\nu,i} = 1$ for $i > 1$.

The function (6.11) has a remarkable property. Dropping the conditions for $|\vec{\alpha}_\mu|$, which is equivalent to summing over all sets $\vec{j} = (j_1, j_2, \dots, j_a)$, gives

$$\sum_{\vec{j}} \omega_a(\vec{n}; j_1, j_2, \dots, j_a) = F_{n_1+1}^{(a)} F_{n_2+1}^{(a)} \dots F_{n_d+1}^{(a)}, \quad (6.19)$$

with the generalized Fibonacci numbers:

$$F_n^{(k)} = \sum_{\alpha=1}^k F_{n-\alpha}^{(k)}, \quad F_1^{(k)} = F_2^{(k)} = 1, \quad F_{n \leq 0}^{(k)} = 0. \quad (6.20)$$

The weighting (6.10) depends on the indices of the MZVs but not on their order, except for n_1 . Furthermore $\omega_{a,1}(\vec{n}; j_1, j_2, \dots, j_a; j'_1)$ is equivalent to $\omega_a(\vec{n}; j_1, j_2, \dots, j_a)$ as long as both functions are used as weightings in identical sums of MZVs. A few special cases of the generalized operator product in identity (6.8) are discussed in the following.

Example 2.1: Identity (6.8) with $(a, b) = (2, 3)$ is relevant for the generalized hypergeometric function ${}_3F_2$:

$$I(0)\{I(0)^{j_1}, I(0, 0)^{j_2}, I(1)^{j_3}, I(1, 0)^{j_4}, I(1, 0, 0)^{j_5}\}I(1)\big|_{z=1} = \sum_{\substack{w=j_1+2j_2+j_3+2j_4+3j_5+2 \\ d=j_3+j_4+j_5+1}} \zeta(\vec{n}) \omega_{2,3}(\vec{n}' - \vec{1}; j_2, j_4, j_5) , \quad (6.21)$$

with

$$\omega_{2,3}(\vec{n}; j_x, j_y, j_z) = \sum_{\substack{\vec{\beta} + \vec{\gamma} \leq \vec{1}; \beta_1 = \gamma_1 = 0 \\ |\vec{\beta}| = j_y; |\vec{\gamma}| = j_z}} \omega_2(\vec{n} - \vec{\beta} - 2\vec{\gamma}; j_x) , \quad (6.22)$$

and

$$\omega_2(\vec{n}; j) = \sum_{\substack{\vec{\alpha} \leq \lfloor \vec{n}/2 \rfloor \\ |\vec{\alpha}| = j}} \binom{\vec{n} - \vec{\alpha}}{\vec{\alpha}} . \quad (6.23)$$

Eq. (6.19) gives the relation to the Fibonacci numbers $F_n \equiv F_n^{(2)}$:

$$\sum_j \omega_2(\vec{n}; j) = F_{n_1+1} F_{n_2+1} \dots F_{n_d+1} . \quad (6.24)$$

Alternatives to the multi-index sum representation (6.8) are possible for generalized operator products with $a = 1$, i.e. the cases where the weighting needs only to consider the permutations of the $I(1, 0, \dots, 0)$ operators. As a result the weighting depends on d_j rather than on n_i .

Example 2.2: The simplest nontrivial case is:

$$I(0)\{I(0)^{j_1}, I(1)^{j_2}, I(1, 0)^{j_3}\}I(1)\big|_{z=1} = \sum_{\substack{w=j_1+j_2+2j_3+2 \\ d=j_2+j_3+1}} \zeta(\vec{n}) \binom{d-1-d_1}{j_3} . \quad (6.25)$$

In contrast to $\omega_{1,2}$ the weighting is written as a binomial coefficient without any sums. It can be understood as the number of ways how the operators $I(1, 0)$ are distributed among the indices \vec{n} . This explains the lower line of the binomial coefficient, since the number of operators $I(1, 0)$ is j_3 . The upper line of the binomial coefficient represents the number of integers n_1, \dots, n_d , to which the third argument $I(1, 0)$ can contribute. This is the depth d minus one, because the $I(1)$ of the MZV index n_d is fixed. Furthermore, one has to subtract d_1 , since the third argument $I(1, 0)$ cannot contribute to n_i when $n_{i+1} = 1$.

For example $(j_1, j_2, j_3) = (1, 1, 1)$ gives $w = 6$ and $d = 3$. One MZV with these properties is $\zeta(3, 2, 1)$. Since the $I(1)$ of n_3 is fixed, the third argument $I(1, 0)$ can only come with n_1 and n_2 . The latter is not possible, since otherwise the sequence for n_3 would start with an $I(0)$

and therefore $n_3 \geq 2$, which contradicts $n_3 = 1$. So there is only one way to obtain this MZV with the given arguments: $I(0, 0, (1, 0), 1, 1)$. Parentheses are included to indicate the position of the third argument. This is in accordance with the weighting in (6.25): $\binom{3-1-1}{1} = 1$. The same holds for all other MZVs of weight $w = 6$ and depth $d = 3$, which have $d_1 = 1$: $\zeta(3, 1, 2)$, $\zeta(2, 3, 1)$ and $\zeta(2, 1, 3)$. There is one MZV with $d_1 = 2$, namely $\zeta(4, 1, 1)$. The corresponding weighting is zero, since the third argument cannot contribute to any n_i . The last MZV to consider for the given weight and depth is $\zeta(2, 2, 2)$ with $d_1 = 0$. This one appears twice, since $I(1, 0)$ can contribute both to n_1 and n_2 : $I(0, 1, 0, (1, 0), 1) + I(0, (1, 0), 1, 0, 1)$. It is easy to check, that this example gives indeed

$$\begin{aligned} I(0)\{I(0), I(1), I(1, 0)\}I(1)\big|_{z=1} \\ = \zeta(3, 2, 1) + \zeta(3, 1, 2) + \zeta(2, 3, 1) + \zeta(2, 1, 3) + 2\zeta(2, 2, 2) , \end{aligned} \quad (6.26)$$

in agreement with (6.25).

Example 2.3: A similar relation applies to the case with $I(1, 0, 0)$ instead of $I(1, 0)$:

$$I(0)\{I(0)^{j_1}, I(1)^{j_2}, I(1, 0, 0)^{j_3}\}I(1)\big|_{z=1} = \sum_{\substack{w=j_1+j_2+3j_3+2 \\ d=j_2+j_3+1}} \zeta(\vec{n}) \binom{d-1-d_1-\bar{d}_2}{j_3} . \quad (6.27)$$

The third argument $I(1, 0, 0)$ contributing to n_i implies $n_{i+1} \geq 3$. Therefore, the number of indices n_i which can originate from $I(1, 0, 0)$ (corresponding to the upper line of the binomial coefficient) is the total number d minus one due to n_d . In addition the term $(d_1 + \bar{d}_2)$, representing the number of integers with $n_{i+1} < 3$, has to be subtracted. The first integer n_1 has to be excluded from these considerations simply because there is no preceding integer to which $I(1, 0, 0)$ could contribute. So \bar{d}_2 , which is the number of indices n_i which equal 2, does not count $n_1 = 2$: $\bar{d}_2 = d_2 - \delta_{2, n_1}$. This distinction is not necessary for d_1 , as $n_1 \neq 1$.

Example 2.4: The following relation includes three arguments of the kind $I(1, 0, \dots, 0)$:

$$\begin{aligned} I(0)\{I(0)^{j_1}, I(1)^{j_2}, I(1, 0)^{j_3}, I(1, 0, 0)^{j_4}\}I(1)\big|_{z=1} \\ = \sum_{\substack{w=j_1+j_2+2j_3+3j_4+2 \\ d=j_2+j_3+j_4+1}} \zeta(\vec{n}) \binom{d-1-d_1-j_4}{j_3} \binom{d-1-d_1-\bar{d}_2}{j_4} . \end{aligned} \quad (6.28)$$

The weighting has a similar explanation as the ones in (6.25) and (6.27). The first binomial coefficient counts all identical terms, which follow from the distribution of the third argument. The second binomial coefficient represents the same for the fourth argument. It is easy to check that the cases $j_3 = 0$ and $j_4 = 0$ reproduce (6.27) and (6.25), respectively.

Identities (6.25), (6.27) and (6.28) give more compact weightings than the ones, which follow from (6.8). It is possible to derive these binomial coefficients from the multi-index sums but it is not obvious how to achieve this. Also, note that $j_1 = 0$ in (6.25) and (6.28) is in accordance with eqs. (6.3) and (6.4), respectively.

6.1.3 Identities with sums

All identities discussed so far are sufficient to write all generalized operator products, which appear in section 3 and 5, in terms of MPLs or MZVs.

Example 3.1: For instance, by using eq. (6.25) it is possible to close the gap in the calculation of the hypergeometric function ${}_2F_1$ in section 3.2:

$$\begin{aligned}
\sum_{\alpha} (-1)^{\alpha} I(0) \{I(0)^{j_1-\alpha}, I(1)^{j_2-\alpha}, I(1,0)^{\alpha}\} I(1) \Big|_{z=1} &= \sum_{\alpha} (-1)^{\alpha} \sum_{\substack{w=j_1+j_2+2 \\ d=j_2+1}} \zeta(\vec{n}) \binom{j_2-d_1}{\alpha} \\
&= \sum_{\substack{w=j_1+j_2+2 \\ d=j_2+1}} \zeta(\vec{n}) \sum_{\alpha} (-1)^{\alpha} \binom{j_2-d_1}{\alpha} \\
&= \sum_{\substack{w=j_1+j_2+2 \\ d=j_2+1}} \zeta(\vec{n}) \delta_{j_2,d_1} \\
&= \zeta(j_1+2, \{1\}^{j_2}) .
\end{aligned} \tag{6.29}$$

The sum in the last step disappears, since there is only one set of indices \vec{n} with weight $w = j_1 + j_2 + 2$, depth $d = j_2 + 1$ and j_2 times the index 1. In this simple case it is possible to combine the outer sum over α with the sum of MZVs to obtain a simple expression. However, in (5.19) there are summations over 17 indices (5.20) and performing the evaluation in the same way gives a rather complicated weighting.

Example 3.2: Summations over two indices are already problematic:

$$\begin{aligned}
\sum_{\alpha_1, \alpha_2} (-1)^{\alpha_1+\alpha_2} I(0) \{I(0)^{j_1-\alpha_1-\alpha_2}, I(0)^{j_2-\alpha_1}, I(0,0)^{\alpha_1}, I(1)^{j_3-\alpha_2}, I(1,0)^{\alpha_2}\} I(1) \Big|_{z=1} \\
= \sum_{\alpha_1, \alpha_2} (-1)^{\alpha_1+\alpha_2} \binom{j_1+j_2-2\alpha_1-\alpha_2}{j_2-\alpha_1} \\
\times I(0) \{I(0)^{j_1+j_2-2\alpha_1-\alpha_2}, I(0,0)^{\alpha_1}, I(1)^{j_3-\alpha_2}, I(1,0)^{\alpha_2}\} I(1) \Big|_{z=1} \\
= \sum_{\substack{w=j_1+j_2+j_3+2 \\ d=j_3+1}} \zeta(\vec{n}) \sum_{\alpha_1, \alpha_2} (-1)^{\alpha_1+\alpha_2} \binom{j_1+j_2-2\alpha_1-\alpha_2}{j_2-\alpha_1} \\
\times \omega_{2,2}(\vec{n}' - \vec{1}; j_1+j_2-2\alpha_1-\alpha_2, \alpha_1; j_3-\alpha_2, \alpha_2) .
\end{aligned} \tag{6.30}$$

In the first step (2.19) is used to combine the identical arguments. Applying identity (6.8) in the next step leads to the given weighting. There is no obvious way how to simplify this expression. Both summations in the first line have a form similar to the one on the l.h.s. of the first line of (6.29). Hence, the question arises, whether they can be evaluated as well. The formalism discussed in the following yields indeed a much simpler expression for (6.30).

The general form of the expressions, which are discussed in this section, is

$$\sum_{\alpha} (-1)^{\alpha} \{c_1^{j_1-\alpha}, c_2^{j_2-\alpha}, (c_2 c_1)^{\alpha}, \dots\} , \quad (6.31)$$

i.e. one argument is a product of two others and their indices share the same summation index α . The dots represent additional arguments. Their indices may depend on other summation indices but not on α . Obviously all operator products in (6.31) contain the same number of c_1 's and c_2 's independent of α . Thus identical products arise, not only from single generalized operator products due to dependent arguments, but also from products with different α . The goal is to combine all the identical terms. Due to the factor $(-1)^{\alpha}$ many of these terms cancel, which leads to simplifications in both the generalized operator product and the summation regions. Eventually, it is possible to obtain for (6.31) a compact representation in terms of MZVs, not only for the generalized operator product, but for the complete expression including the sum.

Obviously, products arising from the generalized product of (6.31) can only be identical, if they contain the same number m of sequences $c_2 c_1$. Denoting the sum of all terms, which include m times the sequence $c_2 c_1$, by s_m , the $\alpha = 0$ operator product of (6.31) can be written as:

$$\{c_1^{j_1}, c_2^{j_2}, \dots\} = \sum_{m \geq 0} s_m . \quad (6.32)$$

The sum is over all possible numbers of sequences $c_2 c_1$. The next term ($\alpha = 1$) gives:

$$\{c_1^{j_1-1}, c_2^{j_2-1}, c_2 c_1, \dots\} = \sum_{m \geq 1} m s_m . \quad (6.33)$$

Here the sum starts with $m = 1$, since there is at least one sequence $c_2 c_1$ coming from the third argument. In addition, the summands are weighted by m . The reason is, that some terms appear more than once: the sequences $c_2 c_1$, which come from the first two arguments, can be exchanged with the ones coming from the third argument without changing the product. E.g. for $m = 2$ there are terms of the form $(\dots c_2 c_1 \dots (c_2 c_1) \dots)$, where the inner brackets indicate, that the second sequence comes from the third argument. To all of these products, there is one identical term: $(\dots (c_2 c_1) \dots c_2 c_1 \dots)$. This explains the weight 2 for the case $m = 2$ in (6.33). For general α and m , there are α sequences $c_2 c_1$ coming from the third argument, while the remaining $m - \alpha$ sequences $c_2 c_1$ originate from the first two arguments. This explains the binomial coefficient in the general relation

$$\{c_1^{j_1-\alpha}, c_2^{j_2-\alpha}, (c_2 c_1)^{\alpha}, \dots\} = \sum_{m \geq \alpha} s_m \binom{m}{\alpha} . \quad (6.34)$$

Inserting this expression in (6.31) yields:

$$\sum_{\alpha} (-1)^{\alpha} \{c_1^{j_1-\alpha}, c_2^{j_2-\alpha}, (c_2 c_1)^{\alpha}, \dots\} = \sum_{m \geq 0} s_m \sum_{\alpha=0}^m (-1)^{\alpha} \binom{m}{\alpha}$$

$$= \sum_{m \geq 0} s_m \delta_{m,0} = s_0 . \quad (6.35)$$

Only s_0 is left. This is the sum of all products, which do not include the sequence $c_2 c_1$. Thus, summations of the form (6.31) can be interpreted as restrictions for the sequences of operators, which appear in the non-commutative products.

For expressions with more sums of the form (6.31), eq. (6.35) has to be applied to each of these individually. It appears, that the indices of some arguments include more than one summation index. A compact representation is possible, when all summation indices appear in the first entry of the generalized operator product and therefore all composed arguments contain c_1 :

$$\begin{aligned} \sum_{\vec{\alpha}} (-1)^{|\vec{\alpha}|} \{ c_1^{j_1 - |\vec{\alpha}|}, c_2^{j_2 - \alpha_1}, (c_2 c_1)^{\alpha_1}, c_3^{j_3 - \alpha_2}, (c_3 c_1)^{\alpha_2}, \dots, c_n^{j_n - \alpha_{n-1}}, (c_n c_1)^{\alpha_{n-1}} \} \\ = c_1^{j_1} \{ c_2^{j_2}, c_3^{j_3}, \dots, c_n^{j_n} \} . \end{aligned} \quad (6.36)$$

Applying (6.35) to each summation allows to identify the forbidden sequences $c_2 c_1, c_3 c_1, \dots, c_n c_1$, i.e. only those products remain, in which all c_1 's appear to the left of all other operators c_2, c_3, \dots, c_n . These are the terms on the r.h.s. of (6.36). All dependent arguments and the sums over the corresponding indices are removed. The number of arguments is reduced from $2n - 1$ to $n - 1$.

The identity (6.36) provides an alternative to determine example 3.1. Setting $n = 2$, $c_1 = I(0)$ and $c_2 = I(1)$ gives the relation:

$$\sum_{\alpha} (-1)^{\alpha} I(0) \{ I(0)^{j_1 - \alpha}, I(1)^{j_2 - \alpha}, I(1, 0)^{\alpha} \} I(1) \Big|_{z=1} = I(0)^{j_1+1} I(1)^{j_2+1} . \quad (6.37)$$

This matches eq. (6.29). Also the $n = 3$ version of (6.36) with $c_1 = I(0)$, $c_2 = I(0)$ and $c_3 = I(1)$ can be used to evaluate the sums over α_1 and α_2 in example 3.2. What remains, is:

$$I(0)^{j_1+1} \{ I(0)^{j_2}, I(1)^{j_3} \} I(1) \Big|_{z=1} = \sum_{\substack{w=j_1+j_2+j_3+2 \\ d=j_3+1; \ n_1 \geq j_1+2}} \zeta(\vec{n}) . \quad (6.38)$$

After the sums are removed, the generalized operator product can be written easily in terms of MZVs by using (6.1). Instead of the complicated weighting in (6.30), there is only the additional condition for n_1 in the sum of MZVs.

The formalism used to obtain the important relations (6.35) and (6.36) can be generalized to cases involving general functions $f(\alpha)$ instead of $(-1)^{\alpha}$:

$$\sum_{\alpha \geq \alpha_0} f(\alpha) \{ c_1^{j_1 - \alpha}, c_2^{j_2 - \alpha}, (c_2 c_1)^{\alpha}, \dots \} = \sum_{m \geq \alpha_0} s_m \sum_{\alpha = \alpha_0}^m f(\alpha) \binom{m}{\alpha} . \quad (6.39)$$

Furthermore, the lower bound $\vec{\alpha}_0$ of the summation is kept general. This allows to handle expressions, where the indices of composed arguments are not simply summation indices, but depend on other quantities as well. Eq. (5.19) has indeed the more general form (6.39), since

it contains multinomial coefficients. However, all of them can be removed by using eq. (2.19). On the other hand, the resulting expression involves an increased number of arguments. Hence, in general one has to decide, whether to handle more complicated functions $f(\alpha)$ or to deal with a larger number of arguments in the generalized operator product. The latter turned out to be more appropriate for the relations considered in this paper, because in this case only the factors $(-1)^\alpha$ remain, which allows to apply the advantageous relation (6.35).

The strategy to simplify (5.19) after all multinomial coefficients are removed, is to use shifts in the summation indices to bring as many of the twelve sums (after application of eqs. (5.21)) as possible into the form (6.35). This allows us to identify all forbidden sequences. However, the configuration for the generalized operator product in (5.19) is not as convenient as the one in identity (6.36). In contrast to c_1 in (6.36), there is no argument of the generalized operator product in (5.19) with an index, that includes all indices of summations of the form (6.35). Furthermore, there are arguments in (5.19), whose indices do not depend on indices of summations of the form (6.35) at all. As a consequence of these two issues, we are not able to present (5.19) in terms of a simplified generalized operator product, as it happens in identity (6.36). Instead the corresponding forbidden sequences are used to write generalized operator products directly in terms of MZVs. This is achieved in a similar manner as for the relations in section 6.1.2. First all permutations of integral operators, which consist only of $I(0)$, are counted. Then it is analysed how the contribution of operators, which include $I(1)$, affects the weighting. Thus inner multiple index sums are related to the first step and outer ones to the second step. This is demonstrated on some examples in the following. We start with generalized operator products with only a few arguments and minor deviations from the form in (6.36), to ultimately present an identity, which can be applied to (5.19).

Example 3.3: A simple case to start with is

$$\sum_{\alpha} (-1)^\alpha I(0) \{I(0)^{j_1-\alpha}, I(0)^{j_2-\alpha}, I(0,0)^\alpha, I(1,0)^{j_3}\} I(1) \Big|_{z=1} = \sum_{\substack{w=2+j_1+j_2+2j_3 \\ d=j_3+1}} \zeta(\vec{n}) \omega'_2(\vec{n} - \vec{2}; j_1) , \quad (6.40)$$

with the weighting:

$$\omega'_2(\vec{n}; j) = \sum_{\substack{\vec{\alpha} \leq \vec{n} \\ |\vec{\alpha}|=j}} 1 . \quad (6.41)$$

The arguments⁷, which can contribute to a sequence of operators $I(0)$, are c_1 and c_2 . The sum over α removes all products with the sequence $c_2 c_1$. Therefore, there is only one way to arrange them: all c_1 to the left of all c_2 resulting in the factor of 1 in the multiple index sum (6.41). The sequence of $n_i - 1$ operators $I(0)$ related to n_i starts with one $I(0)$ stemming either from c_3 for $i > 1$ or from the $I(0)$ to the left of the generalized operator product for $i = 1$.

⁷Some of the arguments in the generalized operator products of this and the following examples are identical. Hence, in order to avoid confusion the argument with the index j_i is referred to as c_i .

As a consequence there can be up to $n_i - 2$ arguments c_1 contributing to n_i . This explains the range of the sum in (6.41), when the arguments of ω'_2 given in identity (6.40) are inserted. The additional condition for $|\vec{\alpha}|$ takes the fixed number of factors c_1 into account. Since the argument c_3 is independent of all others, it is irrelevant for the weighting.

Example 3.4: The following relation includes two arguments, which are not related to summations:

$$\sum_{\alpha} (-1)^{\alpha} I(0) \{I(0)^{j_1-\alpha}, I(0)^{j_2-\alpha}, I(0,0)^{\alpha}, I(1)^{j_3}, I(1,0)^{j_4}\} I(1) \Big|_{z=1} = \sum_{\substack{w=2+j_1+j_2+j_3+2j_4 \\ d=j_3+j_4+1}} \zeta(\vec{n}) \omega'_{2a}(\vec{n} - \vec{2}; j_1, j_3) , \quad (6.42)$$

with:

$$\omega'_{2a}(\vec{n}; j_x, j_y) = \sum_{\substack{\vec{\beta} \leq \vec{1} \\ |\vec{\beta}|=j_y; \beta_1=0}} \omega'_2(\vec{n} + \vec{\beta}; j_x) . \quad (6.43)$$

The generalized operator product contains the same operators of the type $I(0, \dots, 0)$ as example 3.3. This is why the inner sum uses ω'_2 . The only difference is the range. It can be either $n_i - 1$ or $n_i - 2$. This depends on whether c_3 or c_4 contribute to n_{i-1} ($i > 1$). Similar to the identity (6.8), this is taken into account by the outer sum over the multi-index $\vec{\beta} = (\beta_1, \dots, \beta_d)$. Identical terms, which follow from the exchange of arguments c_3 and c_4 are counted this way. For $\beta_i = 1$ the argument c_3 contributes to n_{i-1} , while for $\beta_i = 0$ c_4 does. The condition for $|\vec{\beta}|$ is due to the fixed number j_3 of arguments c_3 . Again n_1 is not affected by these discussions, therefore $\beta_1 = 0$.

Example 3.5: Next, there are two sums of the form (6.35):

$$\sum_{\alpha_1, \alpha_2} (-1)^{\alpha_1 + \alpha_2} I(0) \{I(0)^{j_1-\alpha_1-\alpha_2}, I(0)^{j_2-\alpha_1}, I(0,0)^{\alpha_1}, I(0)^{j_3-\alpha_2}, I(0,0)^{\alpha_2}, I(1,0)^{j_4}\} I(1) \Big|_{z=1} = \sum_{\substack{w=2+j_1+j_2+j_3+2j_4 \\ d=j_4+1; n_i \geq 2}} \zeta(\vec{n}) \omega'_3(\vec{n} - \vec{2}; j_2, j_3) , \quad (6.44)$$

with:

$$\omega'_3(\vec{n}; j_x, j_y) = \sum_{\substack{\vec{\alpha} + \vec{\beta} \leq \vec{n} \\ |\vec{\alpha}|=j_x; |\vec{\beta}|=j_y}} \binom{\vec{\alpha} + \vec{\beta}}{\vec{\beta}} . \quad (6.45)$$

The relevant operators of the type $I(0, \dots, 0)$ are c_1 , c_2 and c_3 . The forbidden sequences are $c_2 c_1$ and $c_3 c_1$. So all c_1 's have to appear to the left of all the c_2 's and c_3 's contributing to the same n_i . The positions of c_2 and c_3 are not completely fixed, since they may be permuted.

With α_i and β_i being the number of factors c_2 and c_3 , respectively, the number of permutations are of course $\binom{\alpha_i + \beta_i}{\alpha_i}$. The range of the sum is the same as in example 3.3, since the operators of the type $I(1, 0, \dots, 0)$ are the same. The additional conditions in the sum in (6.45) are self-explanatory.

Example 3.6: This example includes three sums:

$$\sum_{\alpha_1, \alpha_2, \alpha_3} (-1)^{\alpha_1 + \alpha_2 + \alpha_3} I(0) \{ I(0)^{j_1 - \alpha_1 - \alpha_2}, I(0)^{j_2 - \alpha_1 - \alpha_3}, I(0, 0)^{\alpha_1}, I(0)^{j_3 - \alpha_2}, I(0, 0)^{\alpha_2}, \\ I(1)^{j_4 - \alpha_3}, I(1, 0)^{\alpha_3}, I(1, 0)^{j_5} \} I(1) \Big|_{z=1} = \sum_{\substack{w=j_1+j_2+j_3+j_4+2j_5+2 \\ d=j_4+j_5+1}} \zeta(\vec{n}) \omega'_4 \left(\vec{n} - \vec{2}; \begin{matrix} j_4 \\ j_2, j_3 \end{matrix} \right), \quad (6.46)$$

with

$$\omega'_4 \left(\vec{n}; \begin{matrix} j_x \\ j_a, j_b \end{matrix} \right) = \sum_{\substack{\vec{\mu} \leq \vec{1} \\ |\vec{\mu}|=j_x; \mu_1=0}} \sum_{\substack{\vec{\alpha} + \vec{\beta} \leq \vec{n} + \vec{\mu} \\ |\vec{\alpha}|=j_a; |\vec{\beta}|=j_b}} \left(\begin{matrix} \vec{\alpha} + \vec{\beta} + \vec{\mu}(\delta_{\vec{\alpha}, 0} - 1) \\ \vec{\alpha} \end{matrix} \right). \quad (6.47)$$

Forbidden products are $c_1 c_2$, $c_1 c_3$ and $c_4 c_2$. Therefore, from all arguments contributing to the same sequence of integral operators $I(0)$, the c_1 's appear to the right of all c_2 's and c_3 's. With α_i and β_i being the numbers of c_2 's and c_3 's, respectively, there are

$$\binom{\alpha_i + \beta_i}{\alpha_i} \quad (6.48)$$

possibilities to arrange given numbers of c_1 's, c_2 's and c_3 's without the sequences $c_1 c_2$ or $c_1 c_3$. In case c_5 appears to the left of these operators, the coefficient (6.48) stays the same. But for c_4 the third forbidden product $c_4 c_2$ has to be respected. For $\alpha_i > 0$ there are

$$\binom{\alpha_i + \beta_i - 1}{\alpha_i} \quad (6.49)$$

possibilities, while for $\alpha_1 = 0$ the coefficient remains as in (6.48). Thus for all α_i there are

$$\binom{\alpha_i + \beta_i + \delta_{\alpha_i, 0} - 1}{\alpha_i} \quad (6.50)$$

permutations of given numbers of c_1 's, c_2 's and c_3 's with c_4 appearing to the left and without the sequences $c_1 c_2$, $c_1 c_3$ and $c_4 c_2$. Through the sums over the multi-index $\vec{\mu} = (\mu_1, \dots, \mu_d)$ both cases are taken into account: $\mu_i = 1$ represents c_4 contributing to n_{i-1} ($i > 1$), which yields the coefficient (6.50) in (6.47). On the other hand $\mu_i = 0$ represents c_5 contributing to n_{i-1} ($i > 1$), which gives the coefficient (6.48). Also the range of the inner sums depends on whether c_4 or c_5 contributes to n_{i-1} . It is $n_i - 1$ for the former and $n_i - 2$ for the latter ($i > 1$). The additional conditions for the sums in (6.47) are there for the same reasons as the ones in the previous examples.

Example 3.7: The following expression involves sums over six indices $\vec{\alpha} = (\alpha_1, \dots, \alpha_6)$:

$$\begin{aligned}
& \sum_{\vec{\alpha}} (-1)^{|\vec{\alpha}|} I(0) \{ I(0)^{j_1 - \alpha_1 - \alpha_2 - \alpha_5}, I(0)^{j_2 - \alpha_3 - \alpha_4}, I(0)^{j_3 - \alpha_1 - \alpha_3}, I(0)^{j_4 - \alpha_2 - \alpha_4 - \alpha_6}, \\
& I(0, 0)^{\alpha_1}, I(0, 0)^{\alpha_2}, I(0, 0)^{\alpha_3}, I(0, 0)^{\alpha_4}, I(1, 0)^{j_5 - \alpha_5}, I(1, 0, 0)^{\alpha_5}, I(1, 0)^{j_6 - \alpha_6}, \\
& I(1, 0, 0)^{\alpha_6}, I(1, 0)^{j_7} \} I(1) \Big|_{z=1} \\
& = \sum_{\substack{w=2+j_1+j_2+j_3+j_4+2j_5+2j_6+2j_7 \\ d=1+j_5+j_6+j_7}} \zeta(\vec{n}) \omega'_5 \left(\vec{n} - \vec{2}; \begin{matrix} j_5, & j_6 \\ j_1, & j_2, & j_3, & j_4 \end{matrix} \right), \tag{6.51}
\end{aligned}$$

with the weighting

$$\begin{aligned}
& \omega'_5 \left(\vec{n}; \begin{matrix} j_x, & j_y \\ j_a, & j_b, & j_c, & j_d \end{matrix} \right) \\
& = \sum_{\substack{\vec{\mu} + \vec{\nu} \leq \vec{1} \\ |\vec{\mu}| = j_x; \quad |\vec{\nu}| = j_y \\ \alpha_1, \beta_1 = 0}} \sum_{\substack{\vec{\alpha} + \vec{\beta} + \vec{\gamma} + \vec{\delta} = \vec{n} \\ |\vec{\alpha}| = j_a; \quad |\vec{\beta}| = j_b \\ |\vec{\gamma}| = j_c; \quad |\vec{\delta}| = j_d}} \binom{\vec{\alpha} + \vec{\beta} + \vec{\mu}(\delta_{\vec{\alpha}, 0} - 1)}{\vec{\alpha}} \binom{\vec{\gamma} + \vec{\delta} + \vec{\nu}(\delta_{\vec{\delta}, 0} - 1)}{\vec{\delta}}. \tag{6.52}
\end{aligned}$$

The forbidden sequences related to the six summations are c_3c_1 , c_4c_1 , c_3c_2 , c_4c_2 , c_5c_4 and c_6c_1 . The first four sequences exclusively affect the operators $I(0)$ and are therefore relevant for the inner sums. A sequence of operators $I(0)$ consisting of the arguments c_1 , c_2 , c_3 and c_4 without the forbidden sequences has to start with all c_1 and all c_2 . All permutations of these two factors are allowed. Also all permutations of c_3 and c_4 are allowed. Hence, there are

$$\binom{\alpha_i + \beta_i}{\alpha_i} \binom{\gamma_i + \delta_i}{\delta_i} \tag{6.53}$$

possibilities to build this sequence with α_i , β_i , γ_i and δ_i being the numbers of factors c_1 , c_2 , c_3 and c_4 , respectively. Some of the forbidden sequences interfere with each other: c_5c_4 with c_4c_1 and c_4c_2 . This means, that products, in which these sequences are combined ($c_5c_4c_1$ and $c_5c_4c_2$), do not just vanish but they appear with a negative sign. In other words, an expression, which simply ignores the sequences c_5c_4 , c_4c_1 and c_4c_2 involves too many products. An elegant way to solve this problem, is to introduce the additional forbidden sequences c_1c_4 and c_2c_4 , which have to be considered, if and only if c_5 contributes to n_i . This is possible because $c_4 = c_1 = c_2$.

The freedom of choosing the forbidden sequences c_2c_1 or c_1c_2 for $c_2 = c_1$ is used in the previous examples to avoid interfering forbidden sequences. However, this manipulation is not possible for the example under consideration.

The coefficient within the inner sums of (6.52) depends on which argument of the type $I(1, 0, \dots, 0)$ contributes to n_{i-1} : c_5 , c_6 or c_7 . The upper bound of the inner sums is $n_i - 2$ in all three cases. For c_7 the coefficient is identical to (6.53). For c_6 those contributions have to be subtracted, which start with c_1 . For $\alpha_i > 0$ the first binomial coefficient in (6.53) changes to (6.49). Using Kronecker deltas this can be written for all α_i as (6.50). For c_5 the second binomial coefficient in (6.53) has to be modified. In case $\delta_i > 0$ there are

$$\binom{\gamma_i - 1 + \delta_i}{\delta_i} \tag{6.54}$$

possible permutations of c_3 and c_4 . Therefore, for all δ_i the second binomial coefficient becomes:

$$\binom{\gamma_i + \delta_{\delta_i,0} - 1 + \delta_i}{\delta_i}. \quad (6.55)$$

All these cases are respected by the summations over the d -dimensional multi-indices $\vec{\mu}$ and $\vec{\nu}$. For $\mu_i = \nu_i = 0$ the binomial coefficients remain as in (6.53), so this represents the contribution of c_7 to n_{i-1} . The first binomial coefficient changes to (6.50) for $\nu_i = 1$, while the second becomes (6.55) for $\mu_i = 1$, thus representing the contributions of c_6 and c_5 , respectively.

Note, that the introduction of the additional forbidden sequences leads to the symmetric form of the binomial coefficients in (6.52), which is in agreement with the symmetry of the corresponding generalized operator product. Of course one of the multiple indices $\vec{\alpha}$, $\vec{\beta}$, $\vec{\gamma}$ and $\vec{\delta}$ can be removed, e.g. to obtain the summation region $\vec{\alpha} + \vec{\beta} + \vec{\gamma} \leq \vec{n}$ for the inner sum. This would however destroy the symmetric form of the binomial coefficients.

Example 3.8: This is the most general relation, which includes the previous examples 3.3–3.7 as special cases:

$$\begin{aligned} & \sum_{\vec{\alpha}=(\alpha_1, \alpha_2, \dots, \alpha_9)} (-1)^{|\vec{\alpha}|} I(0) \{ I(0)^{j_1 - \alpha_1 - \alpha_2 - \alpha_5 - \alpha_7 - \alpha_8}, I(0)^{j_2 - \alpha_3 - \alpha_4}, \\ & I(0)^{j_3 - \alpha_1 - \alpha_3}, I(0)^{j_4 - \alpha_2 - \alpha_4 - \alpha_6 - \alpha_9}, I(0,0)^{\alpha_1}, I(0,0)^{\alpha_2}, I(0,0)^{\alpha_3}, I(0,0)^{\alpha_4}, \\ & I(1,0)^{j_5 - \alpha_5}, I(1,0,0)^{\alpha_5}, I(1,0)^{j_6 - \alpha_6}, I(1,0,0)^{\alpha_6}, I(1)^{j_7 - \alpha_7 - \alpha_9}, I(1,0)^{\alpha_7}, \\ & I(1,0)^{\alpha_9 - \alpha_8}, I(1,0,0)^{\alpha_8}, I(1,0)^{j_8} \} I(1) \Big|_{z=1} \\ &= \sum_{\substack{w=j_1+\dots+j_5+2j_6+2j_7+2j_8+2 \\ d=j_5+j_6+j_7+j_8+1}} \omega'_6 \left(\vec{n} - \vec{2}; \begin{matrix} j_5, j_6, j_7 \\ j_1, j_2, j_3, j_4 \end{matrix} \right), \end{aligned} \quad (6.56)$$

with:

$$\begin{aligned} \omega'_6 \left(\vec{n}; \begin{matrix} j_x, j_y, j_z \\ j_a, j_b, j_c, j_d \end{matrix} \right) &= \sum_{\substack{\vec{\mu} + \vec{\nu} + \vec{\sigma} \leq \vec{1} \\ |\vec{\mu}|=j_x; |\vec{\nu}|=j_y; |\vec{\sigma}|=j_z \\ \mu_1, \nu_1, \sigma_1=0}} \sum_{\substack{\vec{\alpha} + \vec{\beta} + \vec{\gamma} + \vec{\delta} = \vec{n} + \vec{\sigma} \\ |\vec{\alpha}|=j_a; |\vec{\beta}|=j_b \\ |\vec{\gamma}|=j_c; |\vec{\delta}|=j_d}} \\ &\times \binom{\vec{\alpha} + \vec{\beta} + (\vec{\mu} + \vec{\sigma})(\delta_{\vec{\alpha},0} - 1)}{\vec{\alpha}} \binom{\vec{\gamma} + \vec{\delta} + (\vec{\nu} + \vec{\sigma})(\delta_{\vec{\delta},0} - 1)}{\vec{\delta}}. \end{aligned} \quad (6.57)$$

Forbidden sequences are c_3c_1 , c_4c_1 , c_3c_2 , c_4c_2 , c_5c_1 , c_6c_4 , c_7c_1 and c_7c_4 . The arguments of the type $I(0, \dots, 0)$ are the same as in example 3.7. Hence, the number of possibilities (6.53) for a sequence of operators $I(0)$ apply here as well. There are four arguments of the type $I(1, 0, \dots, 0)$: c_5 , c_6 , c_7 and c_8 . The range of the inner sums is $n_i - 1$ for c_7 and $n_i - 2$ for all others. For c_8 the coefficient (6.53) is unaffected. For c_5 the first binomial coefficient changes again to (6.50) and for c_6 the second one changes again to (6.55). For c_7 both forbidden sequences including c_5 and c_6 are combined, so both binomial coefficients change to (6.50) and (6.55), respectively. All these modifications are taken into account in (6.57).

The following relations hold in case the functions on both sides are used as weightings within identical sums of MZVs. These relations provide a consistency check, because they result both from the definitions of the weightings and the corresponding generalized operator products:

$$\begin{aligned}
\omega'_6 \left(\vec{n}; \begin{matrix} j_x, j_y, 0 \\ j_a, j_b, j_c, j_d \end{matrix} \right) &= \omega'_5 \left(\vec{n}; \begin{matrix} j_x, j_y \\ j_a, j_b, j_c, j_d \end{matrix} \right) \\
\omega'_6 \left(\vec{n}; \begin{matrix} 0, 0, j_z \\ j_a, j_b, j_c, 0 \end{matrix} \right) &= \omega'_4 \left(\vec{n}; \begin{matrix} j_z \\ j_a, j_b \end{matrix} \right) \\
\omega'_6 \left(\vec{n}; \begin{matrix} 0, 0, 0 \\ j_a, j_b, j_c, 0 \end{matrix} \right) &= \omega'_3(\vec{n}; j_a, j_b) \\
\omega'_6 \left(\vec{n}; \begin{matrix} 0, 0, j_z \\ 0, j_b, j_c, 0 \end{matrix} \right) &= \omega'_{2a}(\vec{n}; j_b, j_z) \\
\omega'_6 \left(\vec{n}; \begin{matrix} 0, 0, 0 \\ 0, j_b, j_c, 0 \end{matrix} \right) &= \omega'_2(\vec{n}; j_b) .
\end{aligned} \tag{6.58}$$

Furthermore the following symmetry holds:

$$\omega'_6 \left(\vec{n}; \begin{matrix} j_x, j_y, j_z \\ j_a, j_b, j_c, j_d \end{matrix} \right) = \omega'_6 \left(\vec{n}; \begin{matrix} j_y, j_x, j_z \\ j_d, j_c, j_b, j_a \end{matrix} \right) . \tag{6.59}$$

6.2 Applications

Two methods to get from expression (3.31) to (3.32) for the hypergeometric function ${}_2F_1$ have been demonstrated in subsection 6.1.3 by using either identity (6.25) or (6.36).

Identity (6.21) allows to express the all-order expansion (3.38) of the $p = 3$ hypergeometric function in terms of MPLs:

$$\begin{aligned}
v_k(z) = & \sum_{l_1+m_1+2(l_2+m_2)+3m_3=k-3} (-1)^{l_1+l_2} \Delta_1^{m_1} \Delta_2^{m_2} \Delta_3^{m_3+1} Q_1^{l_1} Q_2^{l_2} \\
& \times \sum_{\substack{w=k; \ n_1 \geq 3 \\ d=m_1+m_2+m_3+1}} \mathcal{L}i_{\vec{n}}(z) \omega_{2,3}(\vec{n}'' - \vec{1}; l_2, m_2, m_3) , \tag{6.60}
\end{aligned}$$

with $\omega_{2,3}$ defined in (6.22), $n_1'' = n_1 - 2$ and $n_i'' = n_i$ for $i = 2, 3, \dots, d$.

Eq. (6.8) leads to the following representation in terms of MPLs for the coefficient functions (3.45) of ${}_pF_{p-1}$:

$$\begin{aligned}
w_k^p(z) = & \sum_{\vec{l}, \vec{m}} (-1)^{|\vec{l}|} (\Delta_1^p)^{m_1} (\Delta_2^p)^{m_2} \dots (\Delta_{p-1}^p)^{m_{p-1}} (\Delta_p^p)^{m_p+1} (Q_1^p)^{l_1} (Q_2^p)^{l_2} \dots (Q_{p-1}^p)^{l_{p-1}} \\
& \times \sum_{\substack{w=k; \ n_1 \geq p \\ d=m_1+m_2+\dots+m_p+1}} \mathcal{L}i_{\vec{n}}(z) \omega_{p-1,p}(\vec{n}^* - \vec{1}; l_1, l_2, \dots, l_{p-1}; m_1, m_2, \dots, m_p) , \tag{6.61}
\end{aligned}$$

with the weighting $\omega_{p-1,p}$ defined in (6.10), $n_1^* = n_1 - (p - 1)$ and $n_i^* = n_i$ for $i = 2, 3, \dots, d$.

The expression (5.15), which enters the all-order expansions (5.12) and (5.13) of the 5-point open string amplitude, can now be written in terms of MZVs as:

$$\left. \frac{\theta}{\Delta_3} v_k(z) \right|_{z=1} = \sum_{l_1+m_1+2(l_2+m_2)+3m_3=k-3} (-1)^{l_1+l_2} \Delta_1^{m_1} \Delta_2^{m_2} \Delta_3^{m_3} Q_1^{l_1} Q_2^{l_2} \times \sum_{\substack{w=k-1 \\ d=m_1+m_2+m_3+1}} \zeta(\vec{n}) \omega_{2,3}(\vec{n}' - \vec{1}; l_2, m_2, m_3) . \quad (6.62)$$

For the alternative representation (5.16) of the 5-point string amplitude we use eqs. (5.21) in (5.19) to evaluate five of the 17 sums. Some shifts in the remaining 12 summation indices allow to bring nine summations into the form (6.35). Then it is possible to apply the identity (6.56) to arrive at:

$$\begin{aligned} & v(j_1, j_2, j_3, j_4, j_5) \\ &= \sum_{\vec{\beta}, \vec{\delta}, \vec{\epsilon}} (-1)^{|\vec{j}| + \delta_3 + |\vec{\beta}| + \delta_2 + \delta_5 + |\vec{\epsilon}|} I(0) \{ I(0)^{j_1 - \delta_1 - \beta_1 - \beta_2 - \delta_2 - \epsilon_2 - \epsilon_3}, I(0)^{j_2 - \delta_1 - \delta_3 - \beta_3 - \beta_4}, \\ & \quad I(0)^{j_3 - \delta_3 - \delta_4 - \beta_1 - \beta_3}, I(0)^{j_4 - \delta_4 - \delta_5 - \beta_2 - \beta_4 - \epsilon_1}, I(0, 0)^{\beta_1}, I(0, 0)^{\beta_2}, I(0, 0)^{\beta_3}, I(0, 0)^{\beta_4}, \\ & \quad I(1, 0)^{\delta_4 - \epsilon_2}, I(1, 0, 0)^{\epsilon_2}, I(1, 0)^{\delta_1 - \epsilon_1}, I(1, 0, 0)^{\epsilon_1}, I(1)^{j_5 - \delta_5 - \delta_2}, I(1, 0)^{\delta_2}, I(1, 0)^{\delta_5 - \epsilon_3}, \\ & \quad I(1, 0, 0)^{\epsilon_3}, I(1, 0)^{\delta_3} \} I(1) \Big|_{z=1} \\ &= (-1)^{|\vec{j}|} \sum_{\delta_1, \delta_3, \delta_4} (-1)^{\delta_3} \sum_{\substack{w=j_1+j_2+j_3+j_4+j_5+2 \\ d=j_5+\delta_1+\delta_3+\delta_4+1}} \zeta(\vec{n}) \\ & \quad \times \omega'_6 \left(\vec{n} - \vec{2}; \begin{matrix} \delta_4, \delta_1, j_5 \\ j_1 - \delta_1, j_2 - \delta_1 - \delta_3, j_3 - \delta_3 - \delta_4, j_4 - \delta_4, j_5 \end{matrix} \right) \\ &= (-1)^{|\vec{j}|} \sum_{w=j_1+j_2+j_3+j_4+j_5+2} \zeta(\vec{n}) \sum_{\delta_1, \delta_4} (-1)^{d-1-j_5-\delta_1-\delta_4} \\ & \quad \times \omega'_6 \left(\vec{n} - \vec{2}; \begin{matrix} \delta_4, \delta_1, j_5 \\ j_1 - \delta_1, j_2 + j_5 + \delta_4 - d + 1, j_3 + j_5 + \delta_1 - d + 1, j_4 - \delta_4 \end{matrix} \right) . \end{aligned} \quad (6.64)$$

The weighting ω'_6 is defined in (6.57). In the second step the three sums over δ_1 , δ_3 and δ_4 are combined with the sum of MZVs. Apart from the multi-index sums in ω'_6 only two of the sums over the indices (5.20) remain.

The transformations of the generalized operator products to sums of MPLs or MZVs in the all-order expressions for v_k , w_k^p and $v(j_1, j_2, j_3, j_4, j_5)$ as presented in eqs. (6.60) – (6.62) and (6.64) involve weightings, which are rather complicated, due to the variety of multiple index sums and the large number of conditions therein. On one hand the expressions presented in this section have the advantage, that they allow to pick a specific MPL or MZV, respectively, and directly determine the factor they appear with via the weighting. On the other hand the corresponding representations (3.38), (3.45), (5.15) and (5.19) in terms of generalized operator products provide more compact alternatives.

We showed in the previous section that there are identities for generalized operator products, which yield sums of MZV with less complicated weightings, e.g. binomial coefficients. Our

objective in the following is to use such identities for limits of the function $v(j_1, j_2, j_3, j_4, j_5)$ with some of its arguments j_1, j_2, j_3, j_4, j_5 set to zero. Not only is it interesting to see how certain generalized operator products simplify significantly this way, but also will we use these results in the next section to identify particular MZV identities. These limits do not follow directly from (6.64). Going instead one step backwards to the generalized operator product in (6.63) allows to use different identities than (6.56) (in many cases eq. (6.36)). For completeness and as a consistency check we give the MZV representation for all limits. To reduce the number of limits to be calculated we use

$$v(j_1, j_2, j_3, j_4, j_5) = v(j_4, j_3, j_2, j_1, j_5) . \quad (6.65)$$

This symmetry can easily be proven using for instance (5.19) or the symmetry (6.59) of the weighting ω'_6 in (6.64). Those cases, that involve weightings of similar complexity as ω'_6 , i.e. weighting with multiple index sums, are summarized in the appendix.

Four j_i set to zero: We start with the simplest limits, where four of the arguments of $v(j_1, j_2, j_3, j_4, j_5)$ are zero. These calculations are trivial and do not require any identities of section 6.1. No generalized operator products remain and therefore the integral operators can easily be written in terms of MZVs using (3.13). Setting $j_2 = j_3 = j_4 = j_5 = 0$ in (6.63) yields

$$v(j_1, 0, 0, 0, 0) = (-1)^{j_1} I(0)^{j_1+1} I(1) \Big|_{z=1} = (-1)^{j_1} \zeta(j_1 + 2). \quad (6.66)$$

The case $j_1 = j_3 = j_4 = j_5 = 0$ results in the same MZV and we can then use the symmetry (6.65) to determine $v(0, 0, j_3, 0, 0)$ and $v(0, 0, 0, j_4, 0)$:

$$v(j_1, 0, 0, 0, 0) = v(0, j_1, 0, 0, 0) = v(0, 0, j_1, 0, 0) = v(0, 0, 0, j_1, 0) . \quad (6.67)$$

The limit $j_1 = j_2 = j_3 = j_4 = 0$ in (6.63) gives

$$v(0, 0, 0, 0, j_5) = (-1)^{j_5} I(0) I(1)^{j_5+1} \Big|_{z=1} = (-1)^{j_5} \zeta(2, \{1\}^{j_5}) . \quad (6.68)$$

Three j_i set to zero: There are ten different cases with three arguments set to zero. For $j_3 = j_4 = j_5 = 0$ we get:

$$\begin{aligned} v(j_1, j_2, 0, 0, 0) &= (-1)^{j_1+j_2} \sum_{\delta_1} \binom{j_1 + j_2 - 2\delta_1}{j_1 - \delta_1} I(0) \{ I(0)^{j_1+j_2-2\delta_1}, I(1, 0)^{\delta_1} \} I(1) \Big|_{z=1} \\ &= (-1)^{j_1+j_2} \sum_{\delta_1} \binom{j_1 + j_2 - 2\delta_1}{j_1 - \delta_1} \sum_{\substack{w=j_1+j_2+2 \\ d=\delta_1+1; \ n_i \geq 2}} \zeta(\vec{n}) \\ &= (-1)^{j_1+j_2} \sum_{\substack{w=j_1+j_2+2 \\ n_i \geq 2}} \zeta(\vec{n}) \binom{w - 2d}{j_1 + 1 - d} . \end{aligned} \quad (6.69)$$

We used identity (6.2) in the first step and combined the sum over δ_1 with the sum of MZVs in the last line. For $j_2 = j_3 = j_4 = 0$ the $n = 2$ version of identity (6.36) can be used to obtain:

$$v(j_1, 0, 0, 0, j_5) = (-1)^{j_1+j_5} \sum_{\delta_2} (-1)^{\delta_2} I(0) \{ I(0)^{j_1-\delta_2}, I(1)^{j_5-\delta_2}, I(1, 0)^{\delta_2} \} I(1) \Big|_{z=1}$$

$$= (-1)^{j_1+j_5} I(0)^{j_1+1} I(1)^{j_5+1} \Big|_{z=1} = (-1)^{j_1+j_5} \zeta(j_1+2, \{1\}^{j_5}) . \quad (6.70)$$

The same identity applies to the limit $j_2 = j_4 = j_5 = 0$, thus we get:

$$\begin{aligned} v(j_1, 0, j_3, 0, 0) &= (-1)^{j_1+j_3} \sum_{\beta_1} (-1)^{\beta_1} I(0) \{I(0)^{j_1-\beta_1}, I(0)^{j_3-\beta_1}, I(0, 0)^{\beta_1}\} I(1) \Big|_{z=1} \\ &= (-1)^{j_1+j_3} I(0)^{j_1+j_3+1} I(1) \Big|_{z=1} = (-1)^{j_1+j_3} \zeta(j_1+j_3+2) . \end{aligned} \quad (6.71)$$

The calculations for the case $j_2 = j_3 = j_5 = 0$ take the same steps as for $v(j_1, 0, j_3, 0, 0)$, so that:

$$v(j_1, 0, 0, j_4, 0) = v(j_1, 0, j_4, 0, 0) . \quad (6.72)$$

Identity (6.1) is useful for the limit $j_1 = j_3 = j_4 = 0$:

$$\begin{aligned} v(0, j_2, 0, 0, j_5) &= (-1)^{j_2+j_5} I(0) \{I(0)^{j_2}, I(1)^{j_5}\} I(1) \Big|_{z=1} \\ &= (-1)^{j_2+j_5} \sum_{\substack{w=j_2+j_5+2 \\ d=j_5+1}} \zeta(\vec{n}) . \end{aligned} \quad (6.73)$$

Obtaining the MZV representation for $v(0, j_2, j_3, 0, 0)$ requires the identity (6.40) and therefore involves the multi-index sum ω'_2 . The result for $v(0, j_2, j_3, 0, 0)$ is given in (A.1). The limits $v(0, 0, j_3, j_4, 0)$, $v(0, 0, 0, j_4, j_5)$, $v(0, j_2, 0, j_4, 0)$ and $v(0, 0, j_3, 0, j_5)$ follow from eqs. (6.69), (6.70), (6.71) and (6.73), respectively, through the symmetry (6.65).

Two j_i set to zero: There are ten different cases with two j_i set to zero. Setting $j_2 = j_3 = 0$ in (6.63) yields:

$$\begin{aligned} v(j_1, 0, 0, j_4, j_5) &= (-1)^{j_1+j_4+j_5} \sum_{\delta_5, \beta_2, \delta_2, \epsilon_3} (-1)^{\delta_5+\beta_2+\delta_2+\epsilon_3} I(0) \{I(0)^{j_1-\beta_2-\delta_2-\epsilon_2}, I(0)^{j_4-\delta_5-\beta_2}, \\ &\quad I(0, 0)^{\beta_2}, I(1)^{j_5-\delta_5-\delta_2}, I(1, 0)^{\delta_2}, I(1, 0)^{\delta_5-\epsilon_3}, I(1, 0, 0)^{\epsilon_3}\} I(1) \Big|_{z=1} \\ &= (-1)^{j_1+j_4+j_5} \sum_{\delta_5} (-1)^{\delta_5} I(0)^{j_1+1} \{I(0)^{j_4-\delta_5}, I(1)^{j_5-\delta_5}, I(1, 0)^{\delta_5}\} I(1) \Big|_{z=1} \\ &= (-1)^{j_1+j_4+j_5} I(0)^{j_1+j_4+1} I(1)^{j_5+1} \Big|_{z=1} \\ &= (-1)^{j_1+j_4+j_5} \zeta(j_1+j_4+2, \{1\}^{j_5}) . \end{aligned} \quad (6.74)$$

The identity (6.36) had to be applied two times. In the first step with $n = 4$ and in the second with $n = 2$. This limit demonstrates how valuable the identities derived in section 6.1 can be. Starting with four sums of generalized operator products involving seven arguments only a single MZV remains in (6.74). For the limit $j_3 = j_4 = 0$ identity (6.25) leads to

$$\begin{aligned} v(j_1, j_2, 0, 0, j_5) &= (-1)^{j_1+j_2+j_5} \sum_{\delta_1, \delta_2} (-1)^{\delta_1+\delta_2} \binom{j_1+j_2-\delta_1-\delta_2}{j_2-\delta_1} \binom{\delta_2}{\delta_1} I(0) \\ &\quad \times \{I(0)^{j_1+j_2-\delta_1-\delta_2}, I(1)^{j_5+\delta_1-\delta_2}, I(1, 0)^{\delta_2}\} I(1) \Big|_{z=1} \end{aligned}$$

$$\begin{aligned}
&= (-1)^{j_1+j_2+j_5} \sum_{\delta_1=0}^{\min\{j_1, j_2\}} \sum_{\substack{w=j_1+j_2+j_5+2 \\ d=j_5+\delta_1+1}} \zeta(\vec{n}) \omega(j_1+1, j_2+1, d-d_1, \delta_1+1) \\
&= (-1)^{j_1+j_2+j_5} \sum_{\substack{w=j_1+j_2+j_5+2 \\ j_5 < d \leq j_5+1+\min\{j_1, j_2\}}} \zeta(\vec{n}) \omega(j_1+1, j_2+1, d-d_1, d-j_5) , \quad (6.75)
\end{aligned}$$

with the weighting⁸

$$\omega(j_x, j_y, \delta_x, \delta_y) = \binom{\delta_x-1}{\delta_y-1} \binom{j_x+j_y-2\delta_y}{j_x-\delta_y} {}_2F_1 \left[\begin{matrix} \delta_y-j_x, \delta_y-\delta_x \\ 2\delta_y-j_x-j_y \end{matrix}; 1 \right]. \quad (6.76)$$

For $j_3 = j_5 = 0$ we obtain:

$$\begin{aligned}
&v(j_1, j_2, 0, j_4, 0) \\
&= (-1)^{j_1+j_2+j_4} \sum_{\delta_1, \beta_2, \beta_4, \epsilon_1} (-1)^{\beta_2+\beta_4+\epsilon_1} I(0) \{ I(0)^{j_4-\beta_2-\beta_4-\epsilon_1}, I(0)^{j_1-\delta_1-\beta_2}, I(0, 0)^{\beta_2}, \\
&\quad I(0)^{j_2-\delta_1-\beta_4}, I(0, 0)^{\beta_4}, I(1, 0)^{\delta_1-\epsilon_1}, I(1, 0, 0)^{\epsilon_1} \} I(1) \Big|_{z=1} \\
&= (-1)^{j_1+j_2+j_4} \sum_{\delta_1} \binom{j_1+j_2-2\delta_1}{j_1-\delta_1} I(0)^{j_4+1} \{ I(0)^{j_1+j_2-2\delta_1}, I(1, 0)^{\delta_1} \} I(1) \Big|_{z=1} \\
&= (-1)^{j_1+j_2+j_4} \sum_{\delta_1} \binom{j_1+j_2-2\delta_1}{j_1-\delta_1} \sum_{\substack{w=j_1+j_2+j_4+2 \\ d=\delta_1+1; \ n_1 \geq j_4+2; \ n_i \geq 2}} \zeta(\vec{n}) \\
&= (-1)^{j_1+j_2+j_4} \sum_{\substack{w=j_1+j_2+j_4+2 \\ n_1 \geq j_4+2; \ n_i \geq 2}} \zeta(\vec{n}) \binom{w-j_4-2d}{j_1-d+1}. \quad (6.77)
\end{aligned}$$

The first step requires the $n = 4$ version of (6.36) and the second step eq. (6.2). Eventually the sum over δ_1 is combined with the sum of MZVs. With $j_2 = j_4 = 0$ eq. (6.63) becomes:

$$\begin{aligned}
&v(j_1, 0, j_3, 0, j_5) \\
&= (-1)^{j_1+j_3+j_5} \sum_{\beta_1, \delta_2} (-1)^{\beta_1+\delta_2} I(0) \{ I(0)^{j_1-\beta_1-\delta_2}, I(0)^{j_3-\beta_1}, I(0, 0)^{\beta_1}, I(1)^{j_5-\delta_2}, I(1, 0)^{\delta_2} \} I(1) \Big|_{z=1} \\
&= (-1)^{j_1+j_3+j_5} I(0)^{j_1+1} \{ I(0)^{j_3}, I(1)^{j_5} \} I(1) \Big|_{z=1} \\
&= (-1)^{j_1+j_3+j_5} \sum_{\substack{w=j_1+j_3+j_5+2 \\ d=j_5+1; \ n_1 \geq j_1+2}} \zeta(\vec{n}). \quad (6.78)
\end{aligned}$$

Here the $n = 3$ version of identity (6.36) is used in the first and (6.1) in the second step. The MZV representations for $v(j_1, j_2, j_3, 0, 0)$ and $v(0, j_2, j_3, 0, j_5)$ include the weightings ω'_3 and ω'_{2a} ,

⁸To write summation regions for sums of MZVs more compact, we set binomial coefficients to zero for negative arguments (cf. fn. 3 on p. 8). This is, however, not sufficient in (6.75), since the hypergeometric function in the weighting (6.76) has singularities in the region, where the binomial coefficients are set to zero. To ensure convergence we introduce explicit bounds of summation for the sum over δ_1 in the third line of (6.75). This leads to the condition $j_5+1 \leq d \leq j_5+1+\min\{j_1, j_2\}$ for the sum of MZVs in the last line of (6.75).

respectively, which involve multi-index sums. These cases can be found in eqs. (A.2) and (A.3). The symmetry (6.65) allows to straightforwardly determine $v(0, j_2, j_3, j_4, 0)$, $v(0, 0, j_3, j_4, j_5)$, $v(j_1, 0, j_3, j_4, 0)$ and $v(0, j_2, 0, j_4, j_5)$ using the results in (A.2), (6.75), (6.77) and (6.78), respectively.

One j_i set to zero: Setting $j_3 = 0$ in (6.63) yields:

$$\begin{aligned}
& v(j_1, j_2, 0, j_4, j_5) \\
&= (-1)^{j_1+j_2+j_4+j_5} \sum_{\delta_1, \delta_2, \beta_2, \beta_4, \delta_5, \epsilon_1, \epsilon_3} (-1)^{\delta_2+\beta_2+\beta_4+\delta_5+\epsilon_1+\epsilon_3} I(0) \{ I(0)^{j_4-\delta_5-\beta_2-\beta_4-\epsilon_1-\epsilon_3}, \\
& \quad I(0)^{j_1-\delta_1-\delta_2-\beta_2}, I(0, 0)^{\beta_2}, I(0)^{j_2-\delta_1-\beta_4}, I(0, 0)^{\beta_4}, I(1)^{j_5-\delta_2-\delta_5}, I(1, 0)^{\delta_5}, I(1, 0)^{\delta_1-\epsilon_1}, \\
& \quad I(1, 0, 0)^{\epsilon_1}, I(1, 0)^{\delta_2-\epsilon_3}, I(1, 0, 0)^{\epsilon_3} \} I(1) \Big|_{z=1} \\
&= (-1)^{j_1+j_2+j_4+j_5} \sum_{\delta_1, \delta_2} (-1)^{\delta_1+\delta_2} \binom{j_1+j_2-\delta_1-\delta_2}{j_1-\delta_2} \binom{\delta_2}{\delta_1} \\
& \quad \times I(0)^{j_4+1} \{ I(0)^{j_1+j_2-\delta_1-\delta_2}, I(1)^{j_5+\delta_1-\delta_2}, I(1, 0)^{\delta_2} \} I(1) \Big|_{z=1} \\
&= (-1)^{j_1+j_2+j_4+j_5} \sum_{\delta_1=0}^{\min\{j_1, j_2\}} \sum_{\substack{w=j_1+j_2+j_4+j_5+2 \\ d=j_5+\delta_1+1; \ n_1 \geq j_4+2}} \zeta(\vec{n}) \omega(j_1+1, j_2+1, d-d_1, \delta_1+1) \\
&= (-1)^{j_1+j_2+j_4+j_5} \sum_{\substack{w=j_1+j_2+j_4+j_5+2; \ n_1 \geq j_4+2 \\ j_5 < d \leq j_5+1+\min\{j_1, j_2\}}} \zeta(\vec{n}) \omega(j_1+1, j_2+1, d-d_1, d-j_5) . \tag{6.79}
\end{aligned}$$

The $n = 5$ version of (6.36) is applied in the first step and identity (6.25) in step two. The weighting ω is defined in (6.76). The weightings in the MZV representations of the limits $j_5 = 0$ and $j_4 = 0$ involve multiple index sums. They are given in eqs. (A.4) and (A.5). The limits $j_1 = 0$ and $j_2 = 0$ follow from eqs. (A.5) and (6.79), respectively, through the symmetry (6.65).

6.3 Identities for MZVs

In subsection 5.2.2 we presented the α' -expansion (5.16) of the 5-point integral F_2 , in which the kinematic part is separated from the MZVs. The latter are summarized in the function $f(j_1, j_2, j_3, j_4, j_5)$. Combining eqs. (5.5) and (5.16) allows to write $f(j_1, j_2, j_3, j_4, j_5)$ as the coefficient function of the series:

$$\begin{aligned}
& \frac{{}_2F_1 \left[\begin{smallmatrix} -s_1, & s_2 \\ 1+s_1 \end{smallmatrix}; 1 \right] {}_2F_1 \left[\begin{smallmatrix} -s_3, & s_4 \\ 1+s_3 \end{smallmatrix}; 1 \right]}{(1+s_1+s_2)(1+s_3+s_4)} {}_3F_2 \left[\begin{smallmatrix} 1+s_1, & 1+s_4, & 1+s_2+s_3-s_5 \\ 2+s_1+s_2, & 2+s_3+s_4 \end{smallmatrix}; 1 \right] \\
&= \sum_{j_1, j_2, j_3, j_4, j_5 \geq 0} s_1^{j_1} s_2^{j_2} s_3^{j_3} s_4^{j_4} s_5^{j_5} f(j_1, j_2, j_3, j_4, j_5) . \tag{6.80}
\end{aligned}$$

The formula for $f(j_1, j_2, j_3, j_4, j_5)$ can be found in eq. (5.17), which essentially is a sum of products of three types of MZVs representing the three generalized hypergeometric functions

on the l.h.s. of (6.80). Two of the factors in (5.17) are given as the object ζ'_{i_1, i_2} , subject to the definition (5.18), which represent MZVs of the kind $\zeta(i_1 + 1, \{1\}^{i_2-1})$ stemming from the hypergeometric functions ${}_2F_1$. These MZVs can be written in terms of single zeta values [15, 37]. The MZVs, which originate from ${}_3F_2$, are contained in the third factor $v(j_1, j_2, j_3, j_4, j_5)$. This function is given in (5.19) in terms of generalized operator products and in (6.64) in the MZV representation. Combining eqs. (5.17) and (6.64) we can write $f(j_1, j_2, j_3, j_4, j_5)$ in terms of MZVs as well:

$$f(j_1, j_2, j_3, j_4, j_5) = (-1)^k \sum_{\vec{l}=(l_1, l_2, l_3, l_4)} \zeta'_{l_1, l_2} \zeta'_{l_3, l_4} \sum_{w=k-|\vec{l}|} \zeta(\vec{n}) \sum_{\delta_1, \delta_4} (-1)^{d-1-j_5-\delta_1-\delta_4} \\ \times \omega'_6 \left(\vec{n} - \vec{2}; \begin{matrix} \delta_4, \delta_1, j_5 \\ j_1-l_1-\delta_1, j_2+j_5-l_2+\delta_4-d+1, j_3+j_5-l_3+\delta_1-d+1, j_4-l_4-\delta_4 \end{matrix} \right), \quad (6.81)$$

with $k = j_1 + j_2 + j_3 + j_4 + j_5 + 2$.

Our motive in presenting the α' -expansion of F_2 in the form (5.16) was to directly extract identities for MZVs. The object $F_2(s_{13}s_{24})^{-1}$, which equals the product of generalized hypergeometric functions on the l.h.s. of eq. (6.80), is invariant w.r.t. cyclic permutations of the kinematic invariants s_1, s_2, s_3, s_4, s_5 . For this symmetry to hold on the r.h.s. of eq. (6.80), the function $f(j_1, j_2, j_3, j_4, j_5)$ has to be invariant w.r.t. cyclic permutations of its arguments:

$$f(j_1, j_2, j_3, j_4, j_5) = f(j_5, j_1, j_2, j_3, j_4) = f(j_4, j_5, j_1, j_2, j_3) \\ = f(j_3, j_4, j_5, j_1, j_2) = f(j_2, j_3, j_4, j_5, j_1). \quad (6.82)$$

This is not trivially fulfilled. Instead identities for MZVs are generated. Since we know the representation (6.81) for $f(j_1, j_2, j_3, j_4, j_5)$ in terms of MZV, these identities can now be analysed. We mentioned a similar symmetry for the 4-point function (5.2), which generates identities for MZVs as well.

The general identities, which follow from (6.81) through eqs. (6.82), are rather complicated due to the multi-index sums appearing in ω'_6 . However, some more interesting families of MZV identities are included. Instead of multi-index sums, they involve known functions such as binomial coefficients and hypergeometric functions ${}_2F_1$. These identities appear for specific limits, where some of the arguments j_1, j_2, j_3, j_4, j_5 are set to zero. Below we present the MZV identities related to the limits we considered for $v(j_1, j_2, j_3, j_4, j_5)$ in section 6.2. Hence we ignore the cases, which yield multi-index sums, i.e. the ones, which satisfy $(j_2 \neq 0) \wedge (j_3 \neq 0)$ (cf. appendix A). Similar to the previous section, we start with limits, where up to four arguments of $f(j_1, j_2, j_3, j_4, j_5)$ equal zero and eventually discuss cases with only one j_i set to zero. This way we see, which families of MZV identities are included in more general ones. Of course all identities follow from (6.81). We will, however, not use eq. (6.81) to compute the limits for $f(j_1, j_2, j_3, j_4, j_5)$, since it is not obvious how the weighting ω'_6 simplifies in many cases. Instead we work with eq. (5.17) and insert the corresponding limits for $v(j_1, j_2, j_3, j_4, j_5)$, which are computed in the previous section. With those expressions for the MZVs, that originate from ${}_3F_2$, already given, it is straightforward to combine them in eq. (5.17) with the contributions of the hypergeometric functions ${}_2F_1$. This step is particular trivial in case the condition $[(j_1 = 0) \vee (j_2 = 0)] \wedge [(j_3 = 0) \vee (j_4 = 0)]$ holds for $f(j_1, j_2, j_3, j_4, j_5)$. Eq. (5.17) then becomes

$f(j_1, j_2, j_3, j_4, j_5) = v(j_1, j_2, j_3, j_4, j_5)$. Thus, instead of giving explicit expressions for limits of $f(j_1, j_2, j_3, j_4, j_5)$, we will directly present the corresponding MZV identities.

Setting $s_i = 0$ on the r.h.s. of eq. (6.80) gives non-zero contributions for $j_i = 0$ only. Therefore the generating function of MZV identities appearing for $j_i = 0$ can be obtained by setting $s_i = 0$ on the l.h.s. of (6.80).

Similar to the symmetry (6.65) we have

$$f(j_1, j_2, j_3, j_4, j_5) = f(j_4, j_3, j_2, j_1, j_5) . \quad (6.83)$$

It can be seen on the l.h.s. of eq. (6.80), that the corresponding replacements

$$(s_1, s_2, s_3, s_4, s_5) \rightarrow (s_4, s_3, s_2, s_1, s_5)$$

vary only the hypergeometric functions ${}_2F_1$. As a consequence the symmetry (6.83) generates solely the MZV identity (5.3). In contrast to that the cyclic symmetry (6.82) generates more interesting identities, as can be seen in the following.

Four j_i set to zero: There are five different limits for the simplest case:

$$f(j_1, 0, 0, 0, 0) = f(0, j_1, 0, 0, 0) = f(0, 0, j_1, 0, 0) = f(0, 0, 0, j_1, 0) = f(0, 0, 0, 0, j_1) . \quad (6.84)$$

According to eqs. (6.66) – (6.68) the first 4 terms in (6.84) are identical, while $f(0, 0, 0, 0, j_1)$ contains a different MZV. Hence, we obtain the relation

$$\zeta(\alpha_1 + 1) = \zeta(2, \{1\}^{\alpha_1 - 1}) , \quad \alpha_1 \geq 1 . \quad (6.85)$$

This is an instance of the more general relation (5.3).

Three j_i set to zero: The limits with three arguments j_i set to zero can be divided into two types:

$$f(j_1, j_2, 0, 0, 0) = f(0, 0, j_1, j_2, 0) = f(0, 0, 0, j_1, j_2) = f(j_2, 0, 0, 0, j_1) \quad (6.86)$$

and

$$\begin{aligned} f(j_1, 0, j_3, 0, 0) &= f(0, j_1, 0, j_3, 0) = f(0, 0, j_1, 0, j_3) \\ &= f(j_3, 0, 0, j_1, 0) = f(0, j_3, 0, 0, j_1) . \end{aligned} \quad (6.87)$$

We ignore $f(0, j_1, j_2, 0, 0)$, since it involves multi-index sums (cf. (A.1)). While (6.86) includes three independent relations, only one independent family of MZV identities is generated. Comparing eqs. (6.69) and (6.70) allows to write:

$$\sum_{l_1, l_2 \geq 1} \zeta(l_1 + 1, \{1\}^{l_2 - 1}) \sum_{\substack{w = \alpha_1 + \alpha_2 - l_1 - l_2 \\ n_i \geq 2}} \zeta(\vec{n}) \binom{w - 2d}{\alpha_1 - l_1 - d}$$

$$= \sum_{\substack{w=\alpha_1+\alpha_2 \\ n_i \geq 2}} \zeta(\vec{n}) \binom{w-2d}{\alpha_1-d} - \zeta(\alpha_1+1, \{1\}^{\alpha_2-1}) \quad (6.88)$$

with $\alpha_1, \alpha_2 \geq 1$. Up to the relation (5.3) this identity is invariant w.r.t. $\alpha_1 \leftrightarrow \alpha_2$, which allows us to give the additional restriction $\alpha_1 \geq \alpha_2$ in order to generate less linear dependent identities. For examples at weight $w = 5$ with $\alpha_1 = 3, \alpha_2 = 2$ identity (6.88) yields

$$2\zeta(2)\zeta(3) = 3\zeta(5) + \zeta(2, 3) + \zeta(3, 2) - \zeta(4, 1) \quad (6.89)$$

and at weight $w = 6$ with $\alpha_1 = 4, \alpha_2 = 2$ we get

$$\zeta(3)^2 + 2\zeta(2)\zeta(4) = 4\zeta(6) + \zeta(2, 4) + \zeta(3, 3) + \zeta(4, 2) - \zeta(5, 1) . \quad (6.90)$$

The second type of limits (6.87) yields another class of identities. From eqs. (6.71) and (6.73) follows the well known sum theorem [14] :

$$\zeta(\alpha_1) = \sum_{\substack{w=\alpha_1 \\ d=\alpha_2}} \zeta(\vec{n}) , \quad \alpha_1 > \alpha_2 \geq 1 . \quad (6.91)$$

At weights $w = 5$ and $w = 6$ it includes the following identities:

$$\begin{aligned} \zeta(5) &= \zeta(4, 1) + \zeta(3, 2) + \zeta(2, 3) \\ &= \zeta(3, 1, 1) + \zeta(2, 2, 1) + \zeta(2, 1, 2) \\ &= \zeta(2, 1, 1, 1) , \\ \zeta(6) &= \zeta(5, 1) + \zeta(4, 2) + \zeta(3, 3) + \zeta(2, 4) \\ &= \zeta(4, 1, 1) + \zeta(3, 2, 1) + \zeta(3, 1, 2) + \zeta(2, 2, 2) + \zeta(2, 3, 1) + \zeta(2, 1, 3) \\ &= \zeta(3, 1, 1, 1) + \zeta(2, 2, 1, 1) + \zeta(2, 1, 2, 1) + \zeta(2, 1, 1, 2) \\ &= \zeta(2, 1, 1, 1, 1) . \end{aligned} \quad (6.92)$$

The sum theorem is a particular beautiful identity, because it can be described in one sentence: The sum of all MZVs of given weight w and depth d is independent of d .

Two j_i set to zero: There are two classes of limits:

$$f(0, 0, j_3, j_4, j_5) = f(j_5, 0, 0, j_3, j_4) = f(j_4, j_5, 0, 0, j_3) \quad (6.93)$$

and

$$f(j_1, j_2, 0, j_4, 0) = f(j_4, 0, j_1, j_2, 0) = f(0, j_4, 0, j_1, j_2) = f(j_2, 0, j_4, 0, j_1) . \quad (6.94)$$

We obtain one interesting relation via (6.93) with eqs. (6.74) and (6.75):

$$\sum_{l_1, l_2 \geq 1} \zeta(l_1 + 1, \{1\}^{l_2-1}) \sum_{\substack{w=\alpha_1+\alpha_2+\alpha_3-l_1-l_2 \\ \alpha_3 < d \leq \alpha_3 + \min\{\alpha_1-l_1, \alpha_2-l_2\}}} \zeta(\vec{n}) \omega(\alpha_1-l_1, \alpha_2-l_2, d-d_1, d-\alpha_3)$$

$$= \sum_{\substack{w=\alpha_1+\alpha_2+\alpha_3 \\ \alpha_3 < d \leq \alpha_3 + \min\{\alpha_1, \alpha_2\}}} \zeta(\vec{n}) \omega(\alpha_1, \alpha_2, d-d_1, d-\alpha_3) - \zeta(\alpha_2 + \alpha_3 + 1, \{1\}^{\alpha_1-1}) , \quad (6.95)$$

with $\alpha_1, \alpha_2 \geq 1$, $\alpha_3 \geq 0$. This identity contains a hypergeometric function ${}_2F_1$ through the function ω , given in (6.76). Examples are

$$\begin{aligned} \zeta(2)\zeta(2, 1) &= \zeta(2, 3) + \zeta(3, 2) + \zeta(4, 1) + \zeta(2, 1, 2) + \zeta(2, 2, 1) , \\ \zeta(3)\zeta(2, 1) + \zeta(2)\zeta(3, 1) &= \zeta(2, 4) + \zeta(3, 3) + \zeta(4, 2) + 3\zeta(5, 1) + \zeta(2, 1, 3) \\ &\quad + \zeta(2, 3, 1) + \zeta(3, 1, 2) + \zeta(3, 2, 1) - \zeta(4, 1, 1) , \end{aligned} \quad (6.96)$$

for $(\alpha_1, \alpha_2, \alpha_3) = (2, 2, 1)$ and $(\alpha_1, \alpha_2, \alpha_3) = (3, 2, 1)$, respectively. From the results (6.77), (6.78) and eqs. (6.94) follows the family of MVZ identities:

$$\begin{aligned} \sum_{l_1, l_2 \geq 1} \zeta(l_1 + 1, \{1\}^{l_2-1}) \sum_{\substack{w=\alpha_1+\alpha_2+\alpha_3-l_1-l_2 \\ n_1 \geq \alpha_3+2; \ n_i \geq 2}} \zeta(\vec{n}) \binom{w-\alpha_3-2d}{\alpha_1-l_1-d} \\ = \sum_{\substack{w=\alpha_1+\alpha_2+\alpha_3 \\ n_1 \geq \alpha_3+2; \ n_i \geq 2}} \zeta(\vec{n}) \binom{w-\alpha_3-2d}{\alpha_1-d} - \sum_{\substack{w=\alpha_1+\alpha_2+\alpha_3 \\ n_1 > \alpha_2; \ d=\alpha_1}} \zeta(\vec{n}) , \end{aligned} \quad (6.97)$$

with $\alpha_1, \alpha_2 \geq 1$, $\alpha_3 \geq 0$. Two examples with $(\alpha_1, \alpha_2, \alpha_3) = (2, 2, 1)$ and $(\alpha_1, \alpha_2, \alpha_3) = (2, 2, 1)$, respectively, are

$$\begin{aligned} \zeta(2)\zeta(3) &= 2\zeta(5) - \zeta(4, 1) , \\ \zeta(2)\zeta(4) + \zeta(3)\zeta(2, 1) &= 3\zeta(6) + \zeta(3, 3) - \zeta(5, 1) . \end{aligned} \quad (6.98)$$

The transformation $\alpha_1 \leftrightarrow \alpha_2$ in (6.97) changes only the second sum on the r.h.s. Thus we can write:

$$\sum_{\substack{w=\alpha_3 \\ d=\alpha_1; \ n_1 > \alpha_2}} \zeta(\vec{n}) = \sum_{\substack{w=\alpha_3 \\ d=\alpha_2; \ n_1 > \alpha_1}} \zeta(\vec{n}) , \quad \alpha_1 > \alpha_2 \geq 1 , \quad \alpha_3 \geq \alpha_1 + \alpha_2 . \quad (6.99)$$

This is an interesting generalization of the sum theorem (6.91), which arises for $\alpha_2 = 1$. For MZVs of weights $w = 5$ and $w = 6$ additionally to eqs. (6.92) the independent relations

$$\begin{aligned} \zeta(4, 1) &= \zeta(3, 1, 1) , \\ \zeta(5, 1) &= \zeta(3, 1, 1, 1) , \\ \zeta(5, 1) + \zeta(4, 2) &= \zeta(4, 1, 1) + \zeta(3, 2, 1) + \zeta(3, 1, 2) , \end{aligned} \quad (6.100)$$

are generated. The identity (5.3) arises from (6.99) for $\alpha_3 = \alpha_1 + \alpha_2$. All relations following from (6.99) are contained in (6.97), when using the regions for the parameters $\alpha_1, \alpha_2, \alpha_3$ given below that identity. Alternatively we could use the additional condition $\alpha_1 \geq \alpha_2$ in (6.97) and generate the remaining identities with (6.99).

One j_i set to zero: Finally we discuss the identities generated through

$$f(j_1, j_2, 0, j_4, j_5) = f(j_2, 0, j_4, j_5, j_1) . \quad (6.101)$$

Using the symmetry (6.83) and eq. (6.79) we obtain

$$\begin{aligned} \sum_{l_1, l_2 \geq 1} \zeta(l_1 + 1, \{1\}^{l_2-1}) & \sum_{\substack{w=\alpha_1+\alpha_2+\alpha_3+\alpha_4-l_1-l_2; \ n_1 \geq \alpha_3+2 \\ \alpha_4 < d \leq \alpha_4 + \min\{\alpha_1-l_1, \alpha_2-l_2\}}} \zeta(\vec{n}) \omega(\alpha_1-l_1, \alpha_2-l_2, d-d_1, d-\alpha_4) \\ & - \sum_{\substack{w=\alpha_1+\alpha_2+\alpha_3+\alpha_4; \ n_1 \geq \alpha_3+2 \\ \alpha_4 < d \leq \alpha_4 + \min\{\alpha_1, \alpha_2\}}} \zeta(\vec{n}) \omega(\alpha_1, \alpha_2, d-d_1, d-\alpha_4) , \end{aligned} \quad (6.102)$$

with $\alpha_1, \alpha_2 \geq 1$ and $\alpha_3, \alpha_4 \geq 0$. This combination of MZVs appears on one side of the general identity and the other one can be obtained through the transformation

$$(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \rightarrow (\alpha_4 + 1, \alpha_3 + 1, \alpha_2 - 1, \alpha_1 - 1) .$$

For example the parameters $(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (3, 1, 1, 1)$ lead to

$$\begin{aligned} -\zeta(5, 1) &= \zeta(2)\zeta(2, 1, 1) - \zeta(2, 1, 3) - \zeta(2, 3, 1) - \zeta(3, 1, 2) - \zeta(3, 2, 1) - 2\zeta(4, 1, 1) \\ &\quad - \zeta(2, 1, 1, 2) - \zeta(2, 1, 2, 1) - \zeta(2, 2, 1, 1) . \end{aligned} \quad (6.103)$$

Let us emphasize, that identities related to limits, where less arguments j_i equal zero, include those with more j_i set to zero as special cases. All identities presented in this section can be generated using the single expression (6.80). Moreover, additional identities arise, which are not given explicitly here.

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A Limits of $v(j_1, j_2, j_3, j_4, j_5)$

The results for limits of the function $v(j_1, j_2, j_3, j_4, j_5)$, which involve weightings with multi-index sums are listed below. Note that $(j_2 \neq 0) \wedge (j_3 \neq 0)$ holds for all of them. They follow directly from (6.64) and eqs. (6.58). However, we calculate these limits starting from (6.63) and using proper identities of section 6.1 to check their consistency.

The case $j_1 = j_4 = j_5 = 0$ uses identity (6.40):

$$\begin{aligned} & v(0, j_2, j_3, 0, 0) \\ &= (-1)^{j_2+j_3} \sum_{\delta_3, \beta_3} (-1)^{\delta_3+\beta_3} I(0) \{ I(0)^{j_2-\delta_3-\beta_3}, I(0)^{j_3-\delta_3-\beta_3}, I(0, 0)^{\beta_3}, I(1, 0)^{\delta_3} \} I(1) \Big|_{z=1} \end{aligned}$$

$$\begin{aligned}
&= (-1)^{j_2+j_3} \sum_{\delta_3} (-1)^{\delta_3} \sum_{\substack{w=j_2+j_3+2 \\ d=\delta_3+1}} \zeta(\vec{n}) \omega'_2(\vec{n}-\vec{2}; j_2-\delta_3) \\
&= (-1)^{j_2+j_3} \sum_{w=j_2+j_3+2} \zeta(\vec{n}) (-1)^{d-1} \omega'_2(\vec{n}-\vec{2}; j_2-d+1) .
\end{aligned} \tag{A.1}$$

Setting $j_4 = j_5 = 0$ and applying identity (6.44) yields:

$$\begin{aligned}
&v(j_1, j_2, j_3, 0, 0) \\
&= (-1)^{j_1+j_2+j_3} \sum_{\delta_1, \delta_3} (-1)^{\delta_3} \binom{\delta_1 + \delta_3}{\delta_3} \sum_{\beta_1, \beta_3} (-1)^{\beta_1+\beta_3} I(0) \{ I(0)^{j_3-\delta_3-\beta_1-\beta_3}, I(0)^{j_1-\delta_1-\beta_1}, \\
&\quad I(0, 0)^{\beta_1}, I(0)^{j_2-\delta_1-\delta_3-\beta_3}, I(0, 0)^{\beta_3}, I(1, 0)^{\delta_1+\delta_3} \} I(1) \Big|_{z=1} \\
&= (-1)^{j_1+j_2+j_3} \sum_{\delta_1, \delta_3} (-1)^{\delta_3} \binom{\delta_1 + \delta_3}{\delta_3} \sum_{\substack{w=j_1+j_2+j_3+2 \\ d=\delta_1+\delta_3+1}} \zeta(\vec{n}) \omega'_3(\vec{n}-\vec{2}; j_1-\delta_1, j_2-\delta_1-\delta_3) \\
&= (-1)^{j_1+j_2+j_3} \sum_{w=j_1+j_2+j_3+2} \zeta(\vec{n}) \sum_{\delta_3} (-1)^{\delta_3} \binom{d-1}{\delta_3} \omega'_3(\vec{n}-\vec{2}; j_1+\delta_3-d+1, j_2-d+1) .
\end{aligned} \tag{A.2}$$

Identity (6.42) is useful for the limit $j_1 = j_4 = 0$:

$$\begin{aligned}
&v(0, j_2, j_3, 0, j_5) \\
&= (-1)^{j_2+j_3+j_5} \sum_{\delta_3, \beta_3} (-1)^{\delta_3+\beta_3} I(0) \{ I(0)^{j_2-\delta_3-\beta_3}, I(0)^{j_3-\delta_3-\beta_3}, I(0, 0)^{\beta_3}, I(1)^{j_5}, I(1, 0)^{\delta_3} \} I(1) \Big|_{z=1} \\
&= (-1)^{j_2+j_3+j_5} \sum_{\delta_3} (-1)^{\delta_3} \sum_{\substack{w=j_2+j_3+j_5+2 \\ d=1+j_5+\delta_3}} \zeta(\vec{n}) \omega'_{2a}(\vec{n}-\vec{2}; j_2-\delta_3, j_5) \\
&= (-1)^{j_2+j_3+j_5} \sum_{\substack{w=j_2+j_3+j_5+2 \\ d \geq 1+j_5}} \zeta(\vec{n}) (-1)^{d-1-j_5} \omega'_{2a}(\vec{n}-\vec{2}; j_2+j_5+1-d, j_5) .
\end{aligned} \tag{A.3}$$

For $j_5 = 0$ we use identity (6.51) to obtain:

$$\begin{aligned}
&v(j_1, j_2, j_3, j_4, 0) \\
&= (-1)^{j_1+j_2+j_3+j_4} \sum_{\delta_1, \delta_3, \delta_4, \vec{\beta}, \epsilon_1, \epsilon_2} (-1)^{\delta_3+|\vec{\beta}|+\epsilon_1+\epsilon_2} I(0) \{ I(0)^{j_1-\delta_1-\beta_1-\beta_2-\epsilon_2}, I(0)^{j_2-\delta_1-\delta_3-\beta_3-\beta_4}, \\
&\quad I(0)^{j_3-\delta_3-\delta_4-\beta_1-\beta_3}, I(0)^{j_4-\delta_4-\beta_2-\beta_4-\epsilon_1}, I(0, 0)^{\beta_1}, I(0, 0)^{\beta_2}, I(0, 0)^{\beta_3}, I(0, 0)^{\beta_4}, \\
&\quad I(1, 0)^{\delta_4-\epsilon_2}, I(1, 0, 0)^{\epsilon_2}, I(1, 0)^{\delta_1-\epsilon_1}, I(1, 0, 0)^{\epsilon_1}, I(1, 0)^{\delta_3} \} I(1) \Big|_{z=1} \\
&= (-1)^{j_1+j_2+j_3+j_4} \sum_{\delta_1, \delta_3, \delta_4} (-1)^{\delta_3} \sum_{\substack{w=j_1+j_2+j_3+j_4+2 \\ d=\delta_1+\delta_3+\delta_4+1}} \zeta(\vec{n}) \\
&\quad \times \omega'_5 \left(\vec{n} - \vec{2}; \begin{matrix} \delta_4, \delta_1 \\ j_1-\delta_1, j_2-\delta_1-\delta_3, j_3-\delta_3-\delta_4, j_4-\delta_4 \end{matrix} \right) \\
&= (-1)^{j_1+j_2+j_3+j_4} \sum_{w=j_1+j_2+j_3+j_4+2} \zeta(\vec{n}) \sum_{\delta_1, \delta_4} (-1)^{d-1+\delta_1+\delta_4}
\end{aligned}$$

$$\times \omega'_5 \left(\vec{n} - \vec{2}; \begin{matrix} \delta_4, \delta_1 \\ j_1 - \delta_1, j_2 + \delta_4 - d + 1, j_3 + \delta_1 - d + 1, j_4 - \delta_4 \end{matrix} \right). \quad (\text{A.4})$$

Identity (6.46) allows to determine the MZV representation for the limit $j_4 = 0$:

$$\begin{aligned} & v(j_1, j_2, j_3, 0, j_5) \\ &= (-1)^{j_1+j_2+j_3+j_5} \sum_{\delta_1, \delta_3} (-1)^{\delta_3} \binom{\delta_1 + \delta_3}{\delta_1} \sum_{\beta_1, \beta_3, \delta_2} (-1)^{\beta_1+\beta_3+\delta_2} I(0) \{ I(0)^{j_3-\delta_3-\beta_1-\beta_3}, \\ & \quad I(0)^{j_1-\delta_1-\delta_2-\beta_1}, I(0, 0)^{\beta_1}, I(0)^{j_2-\delta_1-\delta_3-\beta_3}, I(0, 0)^{\beta_3}, I(1)^{j_5-\delta_2}, I(1, 0)^{\delta_2}, \\ & \quad I(1, 0)^{\delta_1+\delta_3} \} I(1) \Big|_{z=1} \\ &= (-1)^{j_1+j_2+j_3+j_5} \sum_{\delta_1, \delta_3} (-1)^{\delta_3} \binom{\delta_1 + \delta_3}{\delta_1} \sum_{\substack{w=j_1+j_2+j_3+j_5+2 \\ d=j_5+\delta_1+\delta_3+1}} \zeta(\vec{n}) \\ & \quad \times \omega'_4 \left(\vec{n} - \vec{2}; \begin{matrix} j_5 \\ j_1 - \delta_1, j_2 - \delta_1 - \delta_3 \end{matrix} \right) \\ &= (-1)^{j_1+j_2+j_3+j_5} \sum_{w=j_1+j_2+j_3+j_5+2} \zeta(\vec{n}) \sum_{\delta_1} (-1)^{\delta_1+d-1-j_5} \binom{d-1-j_5}{\delta_1} \\ & \quad \times \omega'_4 \left(\vec{n} - \vec{2}; \begin{matrix} j_5 \\ j_1 - \delta_1, j_2 + j_5 - d + 1 \end{matrix} \right). \end{aligned} \quad (\text{A.5})$$

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